POPULAR DISTANCES IN 3-SPACE

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ABSTRACT. Let $m(n)$ denote the smallest integer m with the property that any set of n points in Euclidean 3-space has an element such that at most m other elements are equidistant from it. We have that

$$cn^{1/3}\log\log n \le m(n) \le n^{3/5}\beta(n),$$

where $c>0$ is a constant and $\beta(n)$ is an extremely slowly growing function, related to the inverse of the Ackermann function.

1. INTRODUCTION

One of Erdős's favorite problems, raised more than half a century ago [4],[9] was the following. What is the maximum number, $f_d(n)$, of times that the unit distance can occur among n points in Euclidean d-space? In [1], we asked a more general question. Given a set $P=\{p_1,...,p_n\}$ of n points in $\mathbb{R}^d$ and positive real numbers $\alpha_1,...,\alpha_n$, let m_i denote the number of points in P whose distance from p_i is α_i. Determine

$$F_d(n)=\max\sum_{i=1}^{n} m_i,$$

where the maximum is taken over all n-element point sets and all possible choices of the numbers α_i. In an extremal configuration, α_i must be one of the *most "popular"* distances from p_i, i.e., a distance which occurs the largest number of times. Clearly, $F_d(n)\ge 2f_d(n)$ for every d and n.

In the planar case, it seems to be hard to determine the asymptotic behavior of the function $F_2(n)$. It is conjectured that right order of magnitude of both $F_2(n)$ and $f_2(n)$ is $O\left(n^{1+c/\log\log n}\right)$, for a suitable constant $c>0$. However, for $d>2$, we have asymptotically tight estimates [1], [5], [6]:

$$F_3(n)=n^2\left(\frac{1}{4}+o(1)\right),$$

$$F_d(n)=n^2\left(1-\frac{1}{\lfloor d/2\rfloor}+o(1)\right)$$

for every $d\ge 4$. In case $d\ge 4$, the bound for $F_d(n)$ is realized by a well-known construction of H. Lenz: take $\lfloor d/2\rfloor$ pairwise orthogonal circles of radius $1/\sqrt{2}$

through the origin, and place n points on them as evenly distributed as possible. Setting $\alpha_i = 1$ for $i = 1, ..., n$, we obtain that

$$m_i = n\left(1 - \frac{1}{\lfloor d/2 \rfloor} + o(1)\right)$$

for every i. That is, from each point the most popular distance is the unit distance, and it occurs roughly the same number of times.

The construction showing that the bound for $F_3(n)$ can be achieved is less symmetric. Take $\lceil n/2 \rceil$ points, $p_1, ..., p_{\lceil n/2 \rceil}$, on a line l, and place the remaining $\lfloor n/2 \rfloor$ points, $p_{\lceil n/2 \rceil + 1}, ..., p_n$, on a circle C around a point of l, so that the plane of C is orthogonal to l. For every $i \leq \lceil n/2 \rceil$, let α_i be the distance between p_i and C. For $i > \lceil n/2 \rceil$, α_i can be arbitrary. Then $m_i = n\left(\frac{1}{2} + o(1)\right)$ for every $i \leq \lceil n/2 \rceil$, and $m_i < 4$ otherwise. In this case, the *sum* of the number of occurrences of the most popular distances over all points is as large as possible, but for about half of the points even the most popular distances occur at most 4 times.

This leads to the following question. What is the largest number $m = m(n)$, for which there exist points $p_1, ..., p_n$ in $\mathbb{R}^3$ and positive reals $\alpha_1, ..., \alpha_n$ such that $m_i \geq m$ for every i? Equivalently, we can ask:

Problem. *What is the smallest integer $m = m(n)$ with the property that any set of n points in $\mathbb{R}^3$ has an element such that fewer than m other elements are equidistant from it?*

At first glance, it is not even clear that $m(n) = o(n)$ holds. In a properly scaled cubic lattice of n points, from each point there are at least $cn^{1/3} \log\log n$ other points at unit distance [5]. Thus, $m(n) \geq cn^{1/3} \log\log n$ for some positive constant c.

Here we show

Theorem 1. *For every $\epsilon > 0$, we have $m(n) = o(n^{3/5+\epsilon})$.*

We present two simple arguments. The first one uses an easy but perhaps interesting generalization of an old theorem of Kővári, Sós, and Turán [7] to directed graphs. It gives the somewhat weaker bound $m(n) = O(n^{2/3})$ (see section 2). For more extremal problems and results for directed graphs, consult [2].

Our second approach is based on a result of Clarkson et al. [3] on the number of incidences between points and spheres (section 3).

2. A Turán-type result for directed graphs

Let G be a *directed graph* with vertex set $V(G)$ and edge set $E(G) \subseteq V(G) \times V(G)$. We would like to establish an upper bound on the number of edges of G, under the assumption that G does not contain certain so-called *forbidden* subgraphs.

For any disjoint sets $V_1, ..., V_k$, construct a directed graph $R(V_1, ..., V_k) = R$ with vertex set $V(R) = \cup_{i=1}^k V_i$ and edge set

$$E(R) = \bigcup_{i=1}^{k-1} V_i \times V_{i+1}.$$

R is called a $\big(|V_1|, ..., |V_k|\big)$*–road*. A $(1, ..., 1)$–road is a *path*.

Theorem 2. *Let G be a directed graph on n vertices, and let s,t be positive integers. If G contains no $(1,s,t)$-road as a subgraph, then it has a vertex of outdegree at most $c_{s,t}n^{1-1/s}$, where $c_{s,t}>0$ is a constant.*

Proof. For $s=1$ the statement is true, so we can assume that $s \geq 2$. Let v be a vertex of G with minimum outdegree m, and let M denote the set of endpoints of the edges of G emanating from v ($|M|=m$). Let G_v be the subgraph of G with vertex set $V(G_v)=V(G)$, consisting of all edges of G whose starting points belong to M.

Let K denote the number of $(s,1)$-roads in G_v. We clearly have

$$K = \sum_{u \in V(G)} \binom{d^+(u)}{s}, \tag{1}$$

$$\sum_{u \in V(G)} d^+(u) = |E(G_v)| \geq |M|m = m^2, \tag{2}$$

where $d^+(u)$ is the indegree of u in G_v. Using the assumption that G contains no $(1,s,t)$-road, we obtain that every s-tuple of M is the set of starting points of at most $t-1$ $(s,1)$-roads and possibly one other $(s,1)$-road ending at v. Therefore,

$$K \leq t\binom{m}{s} < tm^s. \tag{3}$$

Let V_0 denote set of those vertices u, for which $d^+(u) \geq s$. We can assume that V_0 is not empty, otherwise (2) implies that $m < (sn)^{1/2}$, and we are done. Thus, using (1), (2), and Jensen's inequality, we obtain

$$K \geq \sum_{u \in V_0} \binom{d^+(u)}{s} \geq C_s \sum_{u \in V_0} (d^+(u))^s \geq C_s|V_0|(m^2/|V_0|)^s,$$

where $C_s > 0$ is a constant. A comparison with (3) gives

$$tm^s > C_s|V_0|(m^2/|V_0|)^s,$$

so that

$$m \leq (t/C_s)^{1/s}|V_0|^{1-1/s} \leq (t/C_s)^{1/s}n^{1-1/s},$$

completing the proof.

Return now to the problem described in the Introduction. Let $P=\{p_1,...,p_n\}$ be a set of n points in $\mathbb{R}^3$, and let $\alpha_1,...,\alpha_n$ be positive reals. Assume that for every i, there are at least m elements of P at distance α_i from p_i. Construct a directed graph G on the vertex set $V(G)=P$ by drawing an edge from p_i to p_j if their distance is α_i $(1 \leq i,j \leq n)$.

It is easy to verify that G cannot contain a $(1,3,3)$-road $R(V_1,V_2,V_3)$, otherwise all three elements of V_3 would have to lie on the intersection of three spheres centered at the points of V_2, which is impossible, because these points are not collinear. Thus, we can apply Theorem 2 to conclude that $m=m(n)=O(n^{2/3})$.

2. Incidences between points and spheres

We say that a set of spheres is *in general position*, if no three of them pass through the same circle. Combining the Kővári-Sós-Turán theorem (a weaker form of Theorem 2) with a clever probabilistic argument, Clarkson et al. [3] established the following result.

Theorem [3]**.** *The number of incidences between m spheres in general position and n points in $\mathbb{R}^3$ cannot exceed*

$$C\left(m^{3/4}n^{3/4}\beta(m^3/n)+m+n\right),$$

where $C>0$ is a constant and β is an extremely slowly increasing function, related to the inverse of the Ackermann function.

Proof of Theorem 1. Let $P=\{p_1,...,p_n\}$ be a set of n points in $\mathbb{R}^3$, and let S_i denote a sphere of radius α_i around p_i $(1\le i\le n)$. Suppose that each S_i passes through at least m elements of P. Assume without loss of generality that S_n passes through $p_1,...,p_m$. Since no three points of a sphere are collinear, no three spheres S_i,S_j,S_k $(1\le i<j<k\le m)$ have a circle in common. In other words, $S_1,...,S_m$ are in general position.

Hence, we can apply the last Theorem to spheres $S_1,...,S_m$ and points $p_1,...,p_n$, to conclude that the number of incidences between them is at most

$$C\left(m^{3/4}n^{3/4}\beta(m^3/n)+m+n\right).$$

On the other hand, by our assumption, this number is at least m^2, because each S_i $(1\le i\le m)$ is incident to at least m points. Comparing these two bounds, we obtain $m<10Cn^{3/5}\beta(n)$, which completes the proof.

The above argument shows that finding more than m non-collinear elements in P would lead to a better upper bound on $m(n)$.

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