DECOMPOSITION OF MULTIPLE PACKINGS WITH SUBQUADRATIC UNION COMPLEXITY

JÁNOS PACH AND BARTOSZ WALCZAK

Abstract. Let k be a positive integer and let $\mathcal{X}$ be a *k-fold packing* of infinitely many simply connected compact sets in the plane, that is, assume that every point belongs to at most k sets. Suppose that there is a function $f(n) = o(n^2)$ with the property that any n members of $\mathcal{X}$ surround at most $f(n)$ *holes*, which means that the complement of their union has at most $f(n)$ connected components. We use tools from extremal graph theory and the topological Helly theorem to prove that $\mathcal{X}$ can be decomposed into at most p packings, where p is a constant depending only on k and f.

1. Introduction

The notions of multiple packings and coverings were introduced in a geometric setting independently by Harold Davenport and László Fejes Tóth [9]. In the present note, we will be concerned only with packings. A *k-fold packing* is a family $\mathcal{X}$ of sets with the property that the intersection of any $k + 1$ members of $\mathcal{X}$ is empty. A 1-fold packing is simply called a *packing*. The problem of determining the maximum density of a k-fold packing with congruent copies of a fixed convex body has been extensively studied [10]. For small values of k, it was found that densest k-fold lattice packings in the plane split into k packings [3, 5, 14]. The situation gets more complicated for larger values of k, but in general a k-fold packing of convex bodies that are *fat* (that is, the ratio of their circumradii to their inradii is bounded) can be decomposed into $O(k)$ packings, as was shown by Pach [21]. A simple but interesting corollary of this fact is that any k-fold packing of homothets (uniformly scaled and translated copies) of a convex body in $\mathbb{R}^d$ splits into at most $c_d k$ packings, where the constant c_d depends only on the dimension.

The problem of decomposing a family of sets into packings can be rephrased as a coloring problem for intersection graphs. The *intersection graph* of a family $\mathcal{X}$ of sets is a graph on the vertex set $\mathcal{X}$ in which two vertices are joined by an edge if and only if the corresponding members of $\mathcal{X}$ have nonempty

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intersection. A *proper coloring* of a graph is an assignment of colors to the vertices such that no two adjacent vertices receive the same color. Hence, decomposing a family $\mathcal{X}$ into a small number of packings is equivalent to finding a proper coloring of the intersection graph of $\mathcal{X}$ with a small number of colors. The *chromatic number* of a graph is the minimum number of colors used in a proper coloring. The *clique number* of a graph is the maximum size of a set of pairwise adjacent vertices.

If the intersection graph of $\mathcal{X}$ has clique number at most k, then $\mathcal{X}$ is a k-fold packing, but not necessarily the other way around. However, for axis-aligned boxes in Euclidean space, the two notions coincide: a family is a k-fold packing if and only if the clique number of its intersection graph is at most k. In $\mathbb{R}^2$, Asplund and Grünbaum [1] proved that the intersection graphs of axis-aligned rectangles with clique number k have chromatic number $O(k^2)$. This is equivalent to saying that every k-fold packing of axis-aligned rectangles in the plane can be decomposed into $O(k^2)$ packings. On the other hand, Burling [4] constructed 2-fold packings of axis-aligned boxes in $\mathbb{R}^3$ with arbitrarily large chromatic number. Pawlik et al. [22, 23] provided similar constructions for straight-line segments and many other kinds of geometric sets in the plane with the property that their intersection graphs are triangle-free, but they can have arbitrarily large chromatic number. For a survey on coloring geometric intersection graphs, see [17].

The aim of the present note it to show that for every family $\mathcal{F}$ of geometric objects of "small complexity" (in the sense described later), there exists a function $p(k)$ such that every k-fold packing of the plane by members of $\mathcal{F}$ can be split into $p(k)$ packings.

There are some standard measures of complexity for families of geometric objects, used in bounding the computational complexity of various algorithms in motion planning, computer vision, and geometric transversal theory. A *simple arc* with respect to a finite family $\mathcal{X}$ of sets is a Jordan arc whose interior is entirely contained in or disjoint from every set in $\mathcal{X}$. The *union boundary complexity* of $\mathcal{X}$ is the minimum number of simple arcs whose union is the boundary of $\bigcup\mathcal{X}$. A related measure of complexity is the number of *holes* in $\bigcup\mathcal{X}$, that is, bounded arc-connected components of $\mathbb{R}^2 \smallsetminus \bigcup\mathcal{X}$. For families $\mathcal{X}$ of simply connected compact sets, the number of holes is bounded from above by half of the union boundary complexity. However, for some families of geometric objects, the number of holes in the union can be much smaller than the union boundary complexity.

We prove that for any fixed k, every k-fold packing of simply connected compact sets in the plane, the union of any n of which determines a subquadratic number of holes, can be decomposed into a bounded number of packings.

Theorem 1. *Let $k \in \mathbb{N}$, let $f: \mathbb{N} \to \mathbb{N}$ be a function such that $f(n) = o(n^2)$, and let $\mathcal{F}$ be an infinite family of simply connected compact sets in the plane with the property that every finite subfamily $\mathcal{X}$ of $\mathcal{F}$ determines at most $f(|\mathcal{X}|)$ holes. Then there exists a constant $p = p_f(k)$ such that every k-fold packing by members of $\mathcal{F}$ can be decomposed into p packings.*

It is enough to prove Theorem 1 for k-fold packings that are finite subfamilies of $\mathcal{F}$, as then the general statement follows by a standard compactness argument. Therefore, for the remainder of the paper, every k-fold packing that we consider is assumed to be finite. With this assumption, we will prove the following stronger result.

Theorem 2. *Let k, f, and $\mathcal{F}$ be the same as in the previous theorem. Then there exists a constant $p = p_f(k)$ depending on f and k such that the intersection graph of any finite k-fold packing by members of $\mathcal{F}$ has a vertex of degree smaller than $p_f(k)$.*

One of the earliest results on the union boundary complexity and, hence, for the number of holes was established by Kedem et al. [16]. They proved that the union boundary complexity of every family of n *pseudodiscs*, that is, compact sets in the plane bounded by simple closed curves any two of which share at most 2 points, is $O(n)$. Therefore, our theorems imply that any k-fold packing of pseudodiscs splits into a bounded number $p(k)$ of packings.

Matoušek et al. [19] showed that families of n fat triangles in the plane determine $O(n)$ holes. Efrat and Sharir [8] established a near-linear upper bound for families of fat convex sets with the property that any two of them share at most a bounded number of boundary points. Therefore, our theorems also apply to this case and generalize the planar version of the statement on fat convex sets mentioned in the first paragraph. For further results on the complexity of various kinds of fat objects, consult [2, 6, 7, 18].

Our proof of Theorem 2 is based on a result due to Fox and Pach [11], which asserts that the intersection graphs of finite families of arc-connected sets in the plane with no subgraph isomorphic to $K_{t,t}$ have bounded minimum degree. The bound on $p_f(k)$ it gives for fixed f is more than double exponential in k. Using the probabilistic method (specifically, [24]), Micek and Pinchasi [20] proved independently that the intersection graphs of k-fold packings by geometric objects with linear union boundary complexity have minimum degree $O(k)$.

First, in Section 2, we establish our result in a simple special case—for k-fold packings of pseudodiscs. For the proof of Theorem 2 in its full generality, we need a technical lemma on the number of holes determined by 2-fold packings, which is formulated and proved in Section 3. The proof in the general case is presented in Section 4.

2. The case of pseudodiscs

The members of a family of compact sets in the plane are called *pseudodiscs* if they are bounded by simple closed curves, any two of which share at most two points. We first give a short proof of the following.

Proposition 3. *For every positive integer k, there is a constant $p = p(k)$ such that every k-fold packing of pseudodiscs has a member that intersects fewer than p other members.*

For the proof, we use two well-known results. Let $K_{t,t}$ denote a complete bipartite graph with t vertices in each of its parts. The main idea of the proof of Proposition 3 is to show that the intersection graph of any k-fold packing of pseudodiscs has no subgraph isomorphic to $K_{t,t}$ for t large enough, and then apply the following result.

Theorem 4 (Fox, Pach [11, 12])**.** *For any $t \in \mathbb{N}$, there is a constant $c = c(t)$ with the property that the intersection graph of any finite family of arc-connected sets in the plane with no subgraph isomorphic to $K_{t,t}$ has a vertex of degree smaller than c.*

Note that the assertion is true with $c(t) = t(\log t)^\gamma$ with a suitable constant $\gamma > 0$.

We also make use of the following result, often referred to as Topological Helly Theorem.

Theorem 5 (Helly [13])**.** *For any family of pseudodiscs in which every triple has a point in common, all members have a point in common.*

Proof of Proposition 3. By Theorem 4, it is sufficient to prove that the intersection graph of the pseudodiscs contains no $K_{t,t}$ as a subgraph, for a sufficiently large t, to be specified later. Suppose it does, and consider the t pseudodiscs that belong to the first vertex class. Color a triple of them *red* if they have a point in common, and *blue* otherwise. It follows from Theorem 5 that there are no $k + 1$ pseudodiscs, all of whose triples are red. Otherwise, they would share a point, contradicting our assumption that the pseudodiscs form a k-fold packing. Thus, if t is large enough, by Ramsey's theorem the first vertex class of $K_{t,t}$ has 9 pseudodiscs that form a 2-fold packing. Their intersection graph is planar and hence 4-colorable, so at least 3 of these 9 pseudodiscs must be pairwise disjoint. In the same way, we can choose 3 pairwise disjoint pseudodiscs from the second vertex class of $K_{t,t}$. The 6 pseudodiscs chosen induce a $K_{3,3}$ in the intersection graph. Moreover, their arrangement gives rise to a noncrossing drawing of $K_{3,3}$ in the plane, which is the desired contradiction. $\square$

3. Lower bound on the number of holes

For any set X, let $\Gamma(X)$ denote the family of arc-connected components of X, and let $h(X)$ stand for the number of holes in X, that is, let $h(X) = |\Gamma(\mathbb{R}^2 \setminus X)| - 1$.

Lemma 6. *Let $X_1, \dots, X_N$ be not necessarily distinct compact sets in the plane such that the intersection of any three of them is empty. Let S be the set of points that belong to exactly two of $X_1, \dots, X_N$. It follows that*

$$h\Big(\bigcup_{i=1}^{N} X_i\Big) \geqslant |\Gamma(S)| - \sum_{i=1}^{N} |\Gamma(X_i)| + 1.$$

Proof. Consider a bipartite graph G with vertex set

$$V(G) = \Gamma(S) \cup \bigcup_{i=1}^{N} \Gamma(X_i \setminus S)$$

and edge set

$$E(G) = \Big\{(A, B) \in \Gamma(S) \times \Big(\bigcup_{i=1}^{N} \Gamma(X_i \setminus S)\Big) \colon A \cup B \text{ is arc-connected}\Big\}.$$

Let G_i denote the subgraph of G induced by the vertex set $\Gamma(X_i \cap S) \cup \Gamma(X_i \setminus S)$. The number of connected components of G_i is exactly $|\Gamma(X_i)|$, and thus

$$|E(G_i)| \geqslant |V(G_i)| - |\Gamma(X_i)|.$$

Since each edge of G belongs to exactly one of $G_1, \dots, G_N$, and each arc-connected component of S belongs to exactly two of $X_1, \dots, X_N$, we have

$$|E(G)| \geqslant \sum_{i=1}^{N} \big(|V(G_i)| - |\Gamma(X_i)|\big) = |V(G)| + |\Gamma(S)| - \sum_{i=1}^{N} |\Gamma(X_i)|.$$

The graph G is planar, and the number of holes in $X_1 \cup \ldots \cup X_N$ is at least the number of inner faces in a planar drawing of G. Therefore, by Euler's formula, we have

$$|V(G)| - |E(G)| + h\Big(\bigcup_{i=1}^{N} X_i\Big) \geqslant 1.$$

Putting all together, we obtain

$$|\Gamma(S)| \leqslant \sum_{i=1}^{N} |\Gamma(X_i)| + |E(G)| - |V(G)| \leqslant \sum_{i=1}^{N} |\Gamma(X_i)| + h\Big(\bigcup_{i=1}^{N} X_i\Big) - 1. \quad \square$$

4. Proof of Theorem 2

Like in the proof of Proposition 3, we will show that the intersection graph of any k-fold packing by members of $\mathcal{F}$ has no subgraph isomorphic to $K_{t,t}$ for t large enough, and then apply Theorem 4.

We will use a generalization of Turán's theorem to k-uniform hypergraphs. A *k-uniform hypergraph* H consists of a set of *vertices*, denoted by $V(H)$, and a set of *edges*, denoted by $E(H)$, that are k-element subsets of $V(H)$. An *independent set* in such a hypergraph H is a subset of $V(H)$ that does not entirely contain any edge of H.

Theorem 7 (Katona, Nemetz, Simonovits [15]). *For any $k, m \in \mathbb{N}$, every k-uniform hypergraph with $n \geqslant m$ vertices and fewer than $\binom{n}{k}/\binom{m}{k}$ edges contains an independent set of size m.*

A stronger bound with an easy probabilistic proof has been established by Spencer [25], but for our purposes Theorem 7 will suffice.

Lemma 8. *Let $k \geqslant 2$ and r be positive integers, α be a positive real, and $f\colon \mathbb{N} \to \mathbb{N}$ be a function such that $f(n) = o(n^2)$. Then there is an integer $M = M_f(k, r, \alpha)$ that satisfies the following condition. For any collection of not necessarily distinct compact sets $X_1, \ldots, X_n$ in the plane that form a k-fold packing such that*

1. *$|\Gamma(X_i)| \leqslant r$ for $1 \leqslant i \leqslant n$,*
2. *each member of $\Gamma(X_i)$ is simply connected for $1 \leqslant i \leqslant n$,*
3. *the number of k-tuples of sets in $X_1, \ldots, X_n$ with nonempty intersection is at least $\alpha\binom{n}{k}$,*
4. *$h(X_{i_1} \cup \ldots \cup X_{i_t}) \leqslant f(t)$ for any choice of $X_{i_1}, \ldots, X_{i_t}$,*

we have $n < M$.

Proof. We proceed by induction on k. Suppose $k = 2$. By Lemma 6, we have $h(X_1 \cup \ldots \cup X_n) \geqslant \alpha\binom{n}{2} - rn + 1 > f(n)$ if n is large enough. Hence it is enough to define $M_f(2, r, \alpha)$ to be greater than every n for which $\alpha\binom{n}{2} - rn + 1 \leqslant f(n)$.

Now, suppose $k \geqslant 3$. Since the number of k-tuples of sets in $X_1, \ldots, X_n$ with nonempty intersection is at least $\alpha\binom{n}{k}$, one of $X_1, \ldots, X_n$ belongs to at least $\alpha\binom{n}{k} \cdot \frac{k}{n} = \alpha\binom{n-1}{k-1}$ of these k-tuples. Assume without loss of generality that it is X_n, and let $Y_i = X_i \cap X_n$ for $1 \leqslant i \leqslant n-1$. It follows that the intersection of any k sets in $Y_1, \ldots, Y_{n-1}$ is empty, and the number of $(k-1)$-tuples of sets in $Y_1, \ldots, Y_{n-1}$ with nonempty intersection is at least $\alpha\binom{n-1}{k-1}$. By Lemma 6, we have $|\Gamma(Y_i)| \leqslant 2r + f(2) - 1$ for $1 \leqslant i \leqslant n-1$. For any choice of $Y_{i_1}, \ldots, Y_{i_t}$, since each member of $\Gamma(X_n)$ is simply connected, each hole of $Y_{i_1} \cup \ldots \cup Y_{i_t}$ is a hole of $X_{i_1} \cup \ldots \cup X_{i_t}$ and thus $h(Y_{i_1} \cup \ldots \cup Y_{i_t}) \leqslant f(t)$. Therefore, we can apply the induction hypothesis to $Y_1, \ldots, Y_{n-1}$ to conclude that $n - 1 < M_f(k-1, 2r + f(2) - 1, \alpha)$. Hence it is enough to set $M_f(k, r, \alpha) = M_f(k-1, 2r + f(2) - 1, \alpha) + 1$. □

Lemma 9. *Let $k, \ell \in \mathbb{N}$ and $f\colon \mathbb{N} \to \mathbb{N}$ be a function such that $f(n) = o(n^2)$. Then there is an integer $N = N_f(k, \ell)$ that satisfies the following condition. For any family $\{X_1, \ldots, X_n\}$ of simply connected compact sets in the plane that form a k-fold packing such that*

(1) *no* ℓ *members of* $\{X_1, \ldots, X_n\}$ *are pairwise disjoint,*

(2) $h(X_{i_1} \cup \ldots \cup X_{i_t}) \leqslant f(t)$ *for any choice of* $X_{i_1}, \ldots, X_{i_t}$*,*

we have $n < N$*.*

Proof. We proceed by induction on k. For $k = 1$ the statement is trivial. Thus suppose that $k \geqslant 2$ and the statement holds up to $k-1$. Let H denote the k-uniform hypergraph with $V(H) = \{X_1, \ldots, X_n\}$ and $E(H)$ consisting of the k-tuples of sets with nonempty intersection. Let $m = N_f(k-1, \ell)$ and $\alpha = 1/\binom{m}{k}$. Set $N_f(k, \ell) = \max\{M_f(k, 1, \alpha), m\}$ for M_f as claimed by Lemma 8. If $|E(H)| \geqslant \alpha\binom{n}{k}$, then we can apply Lemma 8 to conclude that $n \leqslant N_f(k, \ell)$. Thus, assume $|E(H)| < \alpha\binom{n}{k}$. Since $n \geqslant m$, it follows from Theorem 7 that H contains an independent set $\mathcal{I}$ of size m. Such $\mathcal{I}$ forms a $(k-1)$-fold packing, so we can apply the induction hypothesis to $\mathcal{I}$ and conclude that $m < N_f(k-1, \ell)$, which is a contradiction. □

Proof of Theorem 2. Let $\mathcal{X}$ be a finite subfamily of $\mathcal{F}$ and G be the intersection graph of $\mathcal{X}$. We show, for a suitable constant $t \in \mathbb{N}$, that G contains no subgraph isomorphic to $K_{t,t}$. Then, by Theorem 4, G contains a vertex of degree smaller than $c(t)$, so that we can set $p_f(k) = c(t)$.

Suppose that G contains an *induced* subgraph isomorphic to $K_{\ell,\ell}$, and let $\mathcal{Y} \subset \mathcal{X}$ be the set of vertices of this subgraph. If ℓ is large enough, then by Lemma 6 and the fact that $\mathcal{Y}$ is a 2-fold packing, we have $h(\bigcup \mathcal{Y}) \geqslant \ell^2 - 2\ell + 1 > f(2\ell)$, which is a contradiction. Therefore, we can assume that G contains no induced subgraph isomorphic to $K_{\ell,\ell}$ for an appropriately chosen $\ell \in \mathbb{N}$.

Let $t = N_f(k, \ell)$ for N_f as claimed by Lemma 9. Suppose for a contradiction that G contains a subgraph isomorphic to $K_{t,t}$. Let $\mathcal{A}$ and $\mathcal{B}$ denote its two vertex classes. At least one of $\mathcal{A}, \mathcal{B}$, say $\mathcal{A}$, contains no independent set (packing) of size ℓ, as otherwise the two independent sets, one in $\mathcal{A}$ and one in $\mathcal{B}$, would induce a subgraph isomorphic to $K_{\ell,\ell}$ in G. Therefore, the assumptions of Lemma 9 are satisfied for $\mathcal{A}$, and we conclude that $|\mathcal{A}| < t$. This contradiction completes the proof of Theorem 2. □

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(János Pach) École Polytechnique Fédérale de Lausanne, Switzerland and Alfred Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary
E-mail address: `pach@cims.nyu.edu`

(Bartosz Walczak) École Polytechnique Fédérale de Lausanne, Switzerland and Theoretical Computer Science Department, Faculty of Mathematics and Computer Science, Jagiellonian University, Kraków, Poland
E-mail address: `walczak@tcs.uj.edu.pl`