Tangled Thrackles

János Pach[1], Radoš Radoičić[2], and Géza Tóth[3]

[1] Ecole Polytechnique Fédérale de Lausanne
pach@renyi.hu
[2] Baruch College, City University of New York
rados.radoicic@baruch.cuny.edu
[3] Rényi Institute of Mathematics, Budapest
geza@renyi.hu

Abstract. A *tangle* is a graph drawn in the plane so that any pair of edges have precisely one point in common, and this point is either an endpoint or a point of tangency. If we allow a third option: the common point may be a proper crossing between the two edges, then the graph is called a *tangled thrackle*. We establish the following analogues of Conway's thrackle conjecture: The number of edges of a tangle cannot exceed its number of vertices, n. We also prove that the number of edges of an x-monotone tangled thrackle with n vertices is at most $n+1$. Both results are tight for $n > 3$. For not necessarily x-monotone tangled thrackles, we have a somewhat weaker, but nearly linear, upper bound.

1 Introduction

A *drawing* of a simple undirected graph G is a mapping f that assigns to each vertex a distinct point in the plane and to each edge uv a simple continuous curve (i.e., a homeomorphic image of a closed interval) connecting $f(u)$ and $f(v)$, not passing through the image of any other vertex. For simplicity, the point $f(u)$ assigned to vertex u is also called a vertex of the drawing, and if it leads to no confusion, it is also denoted by u. In the same vein, the curve assigned to uv is called an *edge* of the drawing and it is also denoted by uv. $V(G)$ and $E(G)$ will stand for the vertex set and edge set of the underlying graph G, as well as of its drawing. Throughout the paper, we assume that no three edges have an interior point in common. Paths and cycles on n vertices will be denoted by P_n and C_n, respectively.

A drawing of G is a *thrackle* if every pair of edges have precisely one point in common, either a common vertex or a *proper* crossing. In other words, in a

J. Pach is supported by NSF grant CCF-08-30272, Hungarian Science Foundation EuroGIGA Grant OTKA NN 102029, and Swiss National Science Foundation Grant 200021-125287/1. R. Radoičić is supported by Hungarian Science Foundation Grant OTKA T 046246. G. Tóth is supported by Hungarian Science Foundation Grants OTKA K 83767 and NN 102029.

At a proper crossing of two edges, one edge passes from one side of the other edge to its other side.

thrackle, every two nonadjacent edges cross exactly once, and adjacent edges do not cross. If it creates no confusion, the underlying abstract graph G is also called a thrackle. In the late sixties, Conway [2], [19], [21] conjectured that every thrackle has at most as many edges as vertices. In spite of considerable efforts, this conjecture is still open. If true, the conjecture would be tight, as any cycle other than C_4 is a thrackle [22]. Lovász, Pach, and Szegedy established the first linear upper bound of $2n - 3$ on the number of edges in a thrackle on n vertices, by proving that (the underlying graph of) every bipartite thrackle is actually planar. This bound has been improved since [3], and the current record of $\frac{167}{117}n < 1.43n$ is due to Fulek and Pach [8]. For related results, see [1], [4], [5], [10], [13], [14], and for applications of thrackles, consult [1], [9].

Assuming the aforementioned conjecture is true, Woodall characterized all thrackles: a graph is a thrackle if and only if it has at most one odd cycle, it contains no C_4, and each of its connected components contains at most one cycle. This reduces Conway's conjecture to verifying that each graph consisting of two even cycles that share a single vertex is not a thrackle [14], [22]. Erdős resolved the conjecture for thrackles drawn by straight-line edges (see [15] for an elegant proof of Perles, and its relation to some classical work on diameters of point sets [11]). Cairns and Nikolayevsky [6] proved that every outerplanar thrackle has at most as many edges as vertices. In [15], Pach and Sterling verified the conjecture for the case of *x-monotone* thrackles, that is, thrackles whose edges are curves that meet every vertical line in at most one point.

Inspired by recent work on the number of tangencies in families of curves in various settings (cf. [7], [16]), we propose two new variants of thrackles. A drawing of a graph is called a *tangle* if every pair of edges have precisely one point in common: either a common vertex or a *touching point (a point of tangency)*. In other words, in a tangle, any two nonadjacent edges touch at exactly one interior point, at which the two edges do not cross. We prove the analogue of Conway's conjecture for this variant.

Theorem 1. *Let $n \geq 3$. The maximum number of edges that a tangle of n vertices can have is n.*

A drawing of a graph is called a *tangled thrackle* if every pair of edges have precisely one point in common: either a common vertex, or a point of tangency, or a *proper crossing* (at which an edge passes from one side of the other edge to the other side). In other words, any two nonadjacent edges of a tangled thrackle either touch exactly once, or cross exactly once.

We conjecture the following.

Conjecture 1. Every tangled thrackle on n vertices has $O(n)$ edges.

We confirm our conjecture in the case of x-monotone drawings. Moreover, in this case we have a sharp bound.

A thrackle is called *outerplanar* if its vertices lie on a circle whose interior contain all other edges.

Theorem 2. *Let $n \geq 4$. The maximum number of edges that an x-monotone tangled thrackle of n vertices can have is $n+1$.*

In the general case, the best upper bound we have is slightly superlinear.

Theorem 3. *Let $tt(n)$ denote the maximum number of edges that a tangled thrackle of n vertices can have. Then we have*

$$\left\lfloor \frac{7n}{6} \right\rfloor \leq tt(n) \leq cn \log^{12} n,$$

for some constant c.

2 Proof of Theorem 1

Our proof of Theorem 1 is based on the fact that cycle C_k is a tangle if and only if $k \in \{3, 4\}$ (see Corollary 2), which stands in sharp contrast to the fact that every cycle, except C_4, is a thrackle.

First, we prove the following lemma.

Lemma 1. *If G is a tangle that contains P_5 or C_4 as a subgraph, then G has no other edges.*

Proof. Let G be a tangle, and let H be its subgraph isomorphic to either P_5 or C_4. Let v_i, $i = 1, \ldots, 5$ denote the vertices of H, and $e_i = v_i v_{i+1}$ denote the edges of H. For $(i, j) \in \{(1, 3), (1, 4), (2, 4)\}$ let t_{ij} denote the point of tangency of e_i and e_j. Note that if $H \cong C_4$, then v_1 and v_5 are identical, and t_{14} is not defined. Let $\tilde{H}$ be the (drawing of the) planar graph, obtained from G by introducing new vertices of degree four at the points of tangency t_{ij}, and defining the edges of $\tilde{H}$ maximal pieces of the edges of G that connect two vertices in $V(\tilde{H})$ and contain no other point from $V(\tilde{H})$. If $H \cong P_5$, then $|V(\tilde{H})| = 8$ and $|E(\tilde{H})| = 10$. Similarly, if $H \cong C_4$, then $|V(\tilde{H})| = 6$ and $|E(\tilde{H})| = 8$. Hence, in both cases, $\tilde{H}$ has four faces.

Given a face f of $\tilde{H}$, let the *border* of f be defined as the the set $B(f)$ of all edges $e_i \in E(\tilde{H})$ that contribute infinitely many points to the boundary of f. We claim that the borders of the four faces of $\tilde{H}$ are precisely

$$\{e_1, e_2, e_3\},\ \{e_1, e_2, e_4\},\ \{e_1, e_3, e_4\},\ \{e_2, e_3, e_4\}. \qquad (*)$$

Indeed, this is trivial in the case $H \cong C_4$, since the tangle drawing of H on the sphere has to be topologically equivalent (on the sphere) to Figure 1(c). If $H \cong P_5$, then let $H' \cong P_4$ be the subgraph of H induced by vertices v_i, $i = 1, \ldots, 4$. The tangle drawing of H' has to be topologically equivalent to either (a) or (b) in Figure 1.

According to the order of v_4, v_5, t_{24}, and t_{14} along the edge e_4, and according to which face of $\tilde{H}$ the vertex v_5 belongs to, we have several cases, depicted in Figure 2. It is easy to check that in each case $\tilde{H}$ has *four* faces, and their borders are the triples listed in (*).

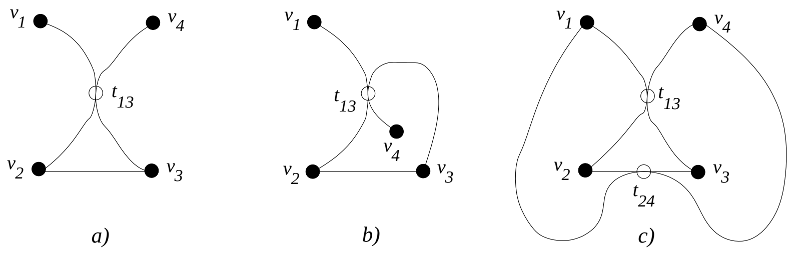

Fig. 1. (a), (b) tangled drawings of P_4; (c) tangled drawing of C_4.

Suppose by contradiction that G contains another edge e. Since G is a tangle, e has precisely one point in common with each edge e_i, $i = 1, \ldots, 4$. On the other hand, e must be contained in a face of $\tilde{H}$. However, according to (*), no border $B(f)$ of a face of $\tilde{H}$ contains all edges e_i, $i = 1, \ldots, 4$. Using our assumption that no three edges have an interior point in common, this is a contradiction. $\square$

The following is an immediate corollary to Lemma 1.

Lemma 2. *Let $k \geq 3$. A cycle C_k is a tangle if and only if $k = 3$ or 4.*

Now we are in a position to complete the proof of Theorem 1. Assume G is a tangle with n vertices and $e \geq n+1$ edges. We can assume G is connected; otherwise, we can find a component of G with more edges than vertices and continue working with it. Note that G cannot contain a C_4 as a subgraph; otherwise, G would contain an additional edge, contradicting Lemma 1. Since G contains at least two more edges than its spanning tree, G has two cycles, C and C'. In view of Lemma 2, C and C' must be triangles. They cannot share an edge; otherwise, G would have a C_4. Since G is connected, there exists a shortest path ℓ (possibly of length 0) between a vertex v of C and a vertex v' of C'. Taking a path of length 2 in C and in C', which starts at v and v', respectively, and connecting them by ℓ, we obtain a copy of P_5 in G. Moreover, the vertices of this path P span at least one additional edge (e.g., the third edge of C that does not belong not to P). This contradicts Lemma 1.

It is easy to see that Theorem 1 is tight for every $n \geq 3$. Indeed, all stars with an additional edge are tangles (see Figure 3: the additional edge can be drawn so that it touches every edge not adjacent to it precisely once).

3 Proof of Theorem 2

Let $G(V, E)$ be an x–monotone tangled thrackle on n vertices. For any vertex v, let $x(v)$ denote the x–coordinate of v. We can also assume that G has no isolated vertex.

In all figures in this paper, vertices marked by empty circles are proper points of tangency, while the original vertices of the graph are represented by black dots.

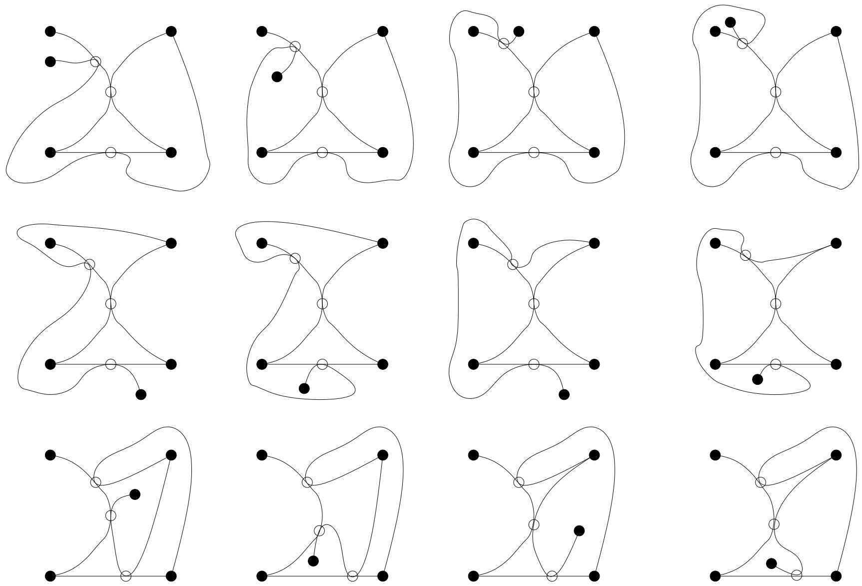

Fig. 2. Tangled drawings of P_5.

Call vertex v of G a *right vertex* (resp. *left vertex*) if for every edge uv incident to it we have $x(u) < x(v)$ (resp. $x(u) > x(v)$). Any vertex that is neither a right vertex nor a left one is said to be *two–sided*. Obviously, G has at most one two–sided vertex. Indeed, if v and v' were two such vertices with $x(v) \leq x(v')$, then any edge whose right endpoint is v would be disjoint from all edges whose left endpoint is v', contradicting the definition of a tangled thrackle.

We distinguish two cases.

Case 1. G has no two–sided vertex.

Among all edges e that share the same left (or right) endpoint v, there is a *highest* edge, that is, one that runs above all other edges e in a small nonempty open interval $(x(v), x(v) + \varepsilon)$. (The *lowest* edge can be defined analogously.)

For each left vertex, delete the highest edge incident to it, and for each right vertex delete the lowest edge. In this way, we removed at most n edges. Suppose that there is a remaining edge uv with $x(u) < x(v)$. Then G must have an edge uu' running above uv, and an edge $v'v$ running below it. Clearly, the edges uu' and $v'v$ cannot have any point in common, contradicting the definition of a tangled thrackle. Therefore, G has at most n edges.

Case 2. G has a two–sided vertex v.

Replace v by two vertices, v_1 and v_2, very close to the original position of v, such that v_1 is to the left of v_2. Slightly modify the drawing of G by reconnecting every edge $uv \in E(G)$ to the vertex v_2 if $x(u) < x(v)$ and to v_1 if $x(u) > x(v)$, in such a way that every edge u_2v_2 crosses all edges v_1u_1, and

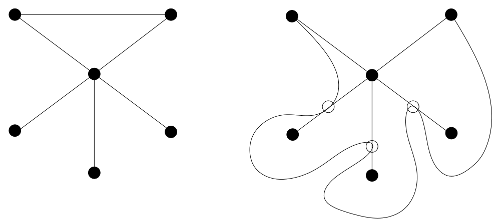

Fig. 3. A star with an additional edge (on the left) and its tangle drawing (on the right).

the resulting drawing G' remains an x-monotone tangled thrackle. G' has $n+1$ vertices, and none of them is two-sided. Therefore, by the previous case, we have $|E(G')| = |E(G)| \leq n+1$, as required.

It remains to prove that Theorem 2 is tight, that is, for every $n \geq 4$ there exist x–monotone tangled thrackles with n vertices and $n+1$ edges.

Lemma 3. *Let G be an x–monotone tangled thrackle, and let uv be an edge of G with $x(u) < x(v)$ which does not touch any other edge. Suppose that uv is the lowest among all edges whose left endpoint is u, and the lowest among all edges whose right endpoint is v. Let G' denote the graph obtained from G by adding two new vertices, u' and v', and replacing the edge uv by the path $uv'u'v$ consisting of the edges uv', $u'v'$, and $u'v$.*

Then G' can also be drawn as an x–monotone tangled thrackle.

Proof. Place u' above u, very close to it, and place v' above v, very close to it. Draw the new edges uv', $u'v'$, and $u'v$ so that
(a) they all run very close to the original edge uv;
(b) they all cross every edge that used to cross uv in G;
(c) every edge whose left endpoint is u crosses both $u'v$ and $u'v'$;
(d) every edge whose right endpoint is v crosses both uv' and $u'v'$. □

A cycle of length 4 with a diagonal can be drawn as an x–monotone tangled thrackle. It has $n = 4$ vertices and $n+1 = 5$ edges. Repeatedly applying Lemma 3 (first with the edge uv, then for uv', say, etc.), for every even $n \geq 6$ we obtain an x–monotone tangled thrackle with n vertices and $n+1$ edges. See Figure 4.

Another construction, suggested by Nikolai Hähnle, is depicted on Figure 5. It consists of a cycle of length 4 with a diagonal uz, plus a number of additional vertices of degree one connected to u.

4 Proof of Theorem 3

Lemma 4. *There are no five curves in the plane with disjoint endpoints such that any two of them have precisely one point in common, a point of tangency, and all of these points are distinct.*

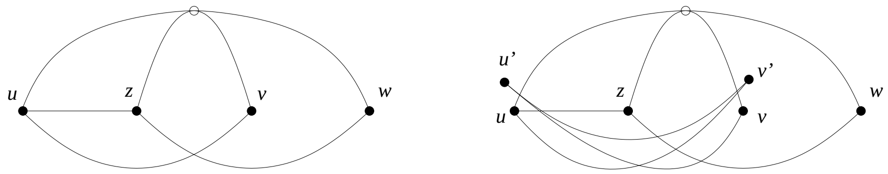

Fig. 4. C_4 with diagonal uz, drawn as an x–monotone tangled thrackle (on the left); edge uv has been replaced by path $uv'u'v$ (on the right).

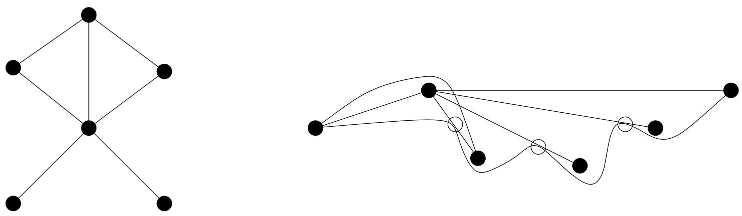

Fig. 5. A graph with n vertices and $n+1$ edges (on the left) and its drawing as an x–monotone tangled thrackle (on the right).

Proof. Suppose there exist five such curves. Fix a different point on each of them, and connect each pair of points using two pieces of the corresponding curves that meet at their point of tangency. This way we obtain a planar drawing of K_5, which may be degenerate in the sense that two adjacent edges may overlap. By slightly perturbing this drawing, if necessary, we can eliminate the common arcs and produce a crossing-free proper drawing of K_5, contradicting Kuratowski's theorem. □

A graph G drawn in the plane so that any two edges have at most one point in common, which is either a common endpoint or a proper crossing (but not a touching point) is called a *simple topological graph.* Two edges of G are said to be *disjoint* if they do not share an endpoint or an interior point. We need the following result from [18].

Lemma 5. [18] *For any $k > 0$, there is a constant c_k such that every simple topological graph with n vertices and no k pairwise disjoint edges has at most $c_k n \log^{4k-8} n$ edges.*

Proof of Theorem 3. Let G be a tangled thrackle with n vertices and more than $c_5 n \log^{12} n$ edges, where $c_5 > 0$ is the constant that appears in Lemma 5.

Slightly modifying the edges of G near their points of tangencies, we can attain that no two edges touch each other, and in the process we do not lose any proper crossings. The resulting drawing is a simple topological graph that has no five pairwise disjoint edges. Indeed, the corresponding five edges of G would be pairwise touching, which contradicts Lemma 4. Thus, the upper bound follows from Lemma 5.

For the lower bound, start with the tangled thrackle drawing of C_6 together with one of its main diagonals, shown in Figure 6. It has the property that there is a vertical line ℓ that intersects every edge exactly once. Pick a point p on ℓ. Using an affine transformation, "squash" this drawing parallel to the direction of the y–axis, to obtain a very "flat" copy of this drawing that lies in a small neighborhood of a horizontal segment. By rotating this drawing about p through $k-1$ different small angles, we can obtain a tangled thrackle. Each copy alone satisfies the conditions, and any pair of edges from different copies cross exactly once. The resulting drawing has $6k$ vertices and $7k$ edges, which proves the lower bound. □

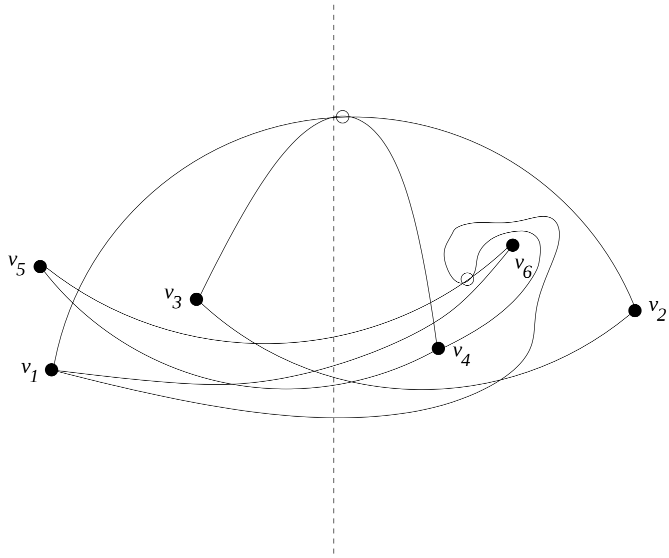

Fig. 6. A tangled thrackle drawing of C_6 with its main diagonal v_1v_4.

Remark. We can modify the notion of tangles and tangled thrackles by allowing several edges to touch one another at the same point.

A drawing of a graph is called a *degenerate tangle* if every pair of edges have precisely one point in common, either a common vertex or a touching point (point of tangency), where several edges may touch one another at the same point. In a *degenerate tangled thrackle*, there is a third option: two edges are also allowed to properly cross each other. It is easy to see that the underlying graph of a degenerate tangle is a planar graph. Therefore, the number of edges of a degenerate tangle of n vertices is at most $3n-6$. Our proof of Theorem

1 breaks down in this case. Not every degenerate tangle can be redrawn as a tangle (consider, for example, a cycle of length four together with one of its main diagonals).

On the other hand, the proof of Theorem 2 goes through without any change for x-monotone degenerate tangled thrackles. It yields that any such graph with n vertices has at most $n + 1$ edges. We believe that a linear upper bound may hold even if we drop the assumption of x-monotonicity.

References

1. E. Ackerman, J. Fox, J. Pach, and A. Suk, On grids in topological graphs, In: *Proceedings of the 25th Annual Symposium on Computational Geometry*, ACM Press, 2009, 403–412.
2. P. Braß, W. Moser, and J. Pach, *Research Problems in Discrete Geometry*, Springer, New York, 2005.
3. G. Cairns and Y. Nikolayevsky, Bounds for generalized thrackles, *Discrete and Computational Geometry*, **23** (2000), 191–206.
4. G. Cairns, M. McIntyre, and Y. Nikolayevsky, The thrackle conjecture for K_5 and $K_{3,3}$, In: *Towards a Theory of Geometric Graphs, Contemp. Math.*, **342**, Amer. Math. Soc., Providence, RI, 2004, 35–54.
5. G. Cairns and Y. Nikolayevsky, Generalized thrackle drawings of non-bipartite graphs, *Discrete and Computational Geometry*, **41** (2009), 119–134.
6. G. Cairns and Y. Nikolayevsky, Outerplanar thrackles, *Graphs and Combinatorics*, **28** (2012), 85–96.
7. J. Fox, F. Frati, J. Pach, and R. Pinchasi, Crossings between curves with many tangencies, In: *Proc. WALCOM: Workshop on Algorithms and Computation*, Lecture Notes in Computer Science, **5942**, Springer-Verlag, 2010, 1–8.
8. R. Fulek and J. Pach, A computational approach to Conway's thrackle conjecture, *Computational Geometry: Theory and Applications*, **44** (2011), 345–355.
9. R. L. Graham, The largest small hexagon, *Journal of Combinatorial Theory, Series A*, **18** (1975), 165–170.
10. J. E. Green and R. D. Ringeisen, Combinatorial drawings and thrackle surfaces, In: *Graph Theory, Combinatorics, and Algorithms, Vol. 2 (Kalamazoo, MI, 1992)*, Wiley-Intersci. Publ., Wiley, New York, 1995, 999–1009.
11. H. Hopf and E. Pannwitz, Aufgabe Nr. 167, *Jahresbericht Deutsch. Math.-Verein.*, **43** (1934), 114.
12. L. Lovász, J. Pach, and M. Szegedy, On Conway's thrackle conjecture, *Discrete and Computational Geometry*, **18** (1998), 369–376.
13. A. Perlstein and R. Pinchasi, Generalized thrackles and geometric graphs in $\mathbb{R}^3$ with no pair of strongly avoiding edges, *Graphs and Combinatorics*, **24** (2008), 373–389.
14. B. L. Piazza, R. D. Ringeisen, and S. K. Stueckle, Subthrackleable graphs and four cycles, In: *Graph Theory and Applications (Hakone, 1990), Discrete Mathematics*, **127** (1994), 265–276.
15. J. Pach and E. Sterling, Conway's conjecture for monotone thrackles, *American Mathematical Monthly*, **118** (2011), 544–548.
16. J. Pach, A. Suk, and M. Treml, Tangencies between families of disjoint regions in the plane, *Computational Geometry: Theory and Applications*, **45** (2012), 131–138.

17. J. Pach and J. Törőcsik, Some geometric applications of Dilworth's theorem, *Discrete and Computational Geometry*, **12** (1994), 1–7.
18. J. Pach and G. Tóth, Disjoint edges in topological graphs, *Journal of Combinatorics*, **1** (2010), 335–344.
19. R. D. Ringeisen, Two old extremal graph drawing conjectures: progress and perspectives, *Congressus Numerantium*, **115** (1996), 91–103.
20. G. Tóth, Note on geometric graphs, *Journal of Combinatorial Theory, Series A*, **89** (2000), 126–132.
21. Unsolved problem. Chairman: P. Erdős, In: *Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972)*, Inst. Math. Appl., Southend-on-Sea, 1972, 351–363.
22. D. R. Woodall, Thrackles and deadlock, In: *Combinatorial Mathematics and Its Applications (Welsh, D. J. A., ed.)*, Academic Press, 1969, 335–347.