Points surrounding the origin

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on the occasion of their sixtieth birthdays.

Abstract

For $d > 2$ and $n > d+1$, let $P = \{p_1, \ldots, p_n\}$ be a set of points in $\mathbb{R}^d$ whose convex hull contains the origin 0 in its interior. We show that if $P \cup \{0\}$ is in general position, then there exists a d-tuple $Q = \{p_{i_1}, \ldots, p_{i_d}\} \subset P$ such that 0 is not contained in the convex hull of $Q \cup \{p\}$ for any $p \in P \setminus Q$. A generalization of this property is also considered.

1 Introduction

Let P be a finite point set in $\mathbb{R}^d$, in *general position* with respect to the origin 0, in the sense that no k elements of $P \cup \{0\}$ lie in a $(k-2)$-flat $(2 \leq k \leq d+1)$. We say that *P surrounds the origin* if for every $Q \subset P$ with $|Q| = d$, there exists an $x \in P \setminus Q$ such that the origin is contained in conv $\{x\} \cup Q$, the convex hull of $\{x\} \cup Q$.

In the special case $d = 2$, consider a planar point set $P = \{p_1, p_2, \ldots, p_n\}$, whose elements are listed and enumerated mod n in the clockwise cyclic order, as they can be seen from the origin. Clearly, P surrounds the origin if and only if for every i, there exists j such that the triangle $p_i p_{i+1} p_j$ contains the origin in its

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interior. In particular, it follows that n must be odd. This so-called "antipodality" property of point sets was explored by Lovász [7] and others [3, 10] to bound the maximum number "halving lines" of a set of n points in the plane.

R. Strausz [11] discovered an interesting property of planar point sets P surrounding the origin: For any coloring of the elements of P with *three* colors such that every color is used at least once, there is a *rainbow* triangle which contains the origin in its interior, that is, a triangle whose vertices are of different colors. Using the terminology of [1], we can say that the 3-uniform hypergraph consisting of all triples in P whose convex hulls contain the origin is *tight*.

It turns out, somewhat counter-intuitively, that in *three* and higher dimensions, there exists no nontrivial point set that surrounds the origin. By counting simplices of triangulations, analogously to the argument in [4], we show the following.

Theorem 1. *Let $d > 2$ and let P be a finite point set in $\mathbb{R}^d$ in general position with respect to the origin, and suppose that $|P| > d+1$. Then P contains d-tuple Q such that the convex hull of $Q \cup \{x\}$ does not contain the origin for any $x \in P$.*

The above property can be generalized as follows. For any $0 \leq k \leq d+1$, we say that the set $P \subset \mathbb{R}^d$ has property $S(k)$, if for every $Q \subset P$ with $|Q| = k$, there exists an $R \subset P \setminus Q$ with $|R| = d+1-k$, such that the origin is contained in conv $Q \cup R$.

Obviously, property $S(k)$ depends on the choice of origin, and it is *monotonic* in the sense that property $S(k)$ is stronger than property $S(k-1)$. Carathéodory's theorem (see [6] or [12]) states that if the origin is contained in conv P, then it is contained in the convex hull of some $(d+1)$-tuple of P, or simply, $0 \in$ conv P implies property $S(0)$. In fact, we may triangulate P from any given point of P which implies that properties $S(0)$ and $S(1)$ are equivalent.

At the other end of the spectrum, it is easy to show that if $|P| > d+1$, then P does *not* have property $S(d+1)$. (This immediately follows by triangulating the point set; see either part of Claim 4.) Theorem 1 tells us that properties $S(d+1)$ and $S(d)$ are equivalent. The following two questions arise.

Problem 2. *Let $d \geq 2$ be fixed.*

1. *What is the largest integer $k = k(d)$ such that there are arbitrarily large finite point sets $P \subset \mathbb{R}^d$ in general position with respect to the origin that have property $S(k)$?*
2. *What is the smallest integer $K = K(d)$ such that there is no finite point set $P \subset \mathbb{R}^d$ in general position with respect to the origin with more than $d+1$ elements, which has property $S(K+1)$?*

Clearly, we have $k(d) \leq K(d)$, for every d.

In Section 2 we give a proof of Theorem 1, and in Section 3 we find an equivalent formulation of Problem 2 in terms of facets of convex polytopes. From this viewpoint it will be easy to extract the following lower bound on $k(d)$.

Theorem 3. *For every integer $d \geq 2$, there exist arbitrarily large point sets in $\mathbb{R}^d$ in general position with respect to the origin with property $S(\lfloor \frac{d}{2} \rfloor + 1)$. In other words, $k(d) \geq \lfloor \frac{d}{2} \rfloor + 1$.*

From Theorems 1 and 3 it follows that $k(2) = 2 = K(2)$ and $k(3) = 2 = K(3)$. It would be interesting to know if there are values of d for which $k(d) < K(d)$ holds.

2 Proof of Theorem 1

Given a finite set of points A in general position in $\mathbb{R}^d$, a *triangulation* of A is a decomposition of $\operatorname{conv} A$, the convex hull of A, into non-overlapping d-dimensional simplices, each of which is spanned by $d+1$ elements of A and contains no other point of A. In the plane, the number of triangles in *any* triangulation of A is the same, but this is not the case in higher dimensions. For a survey, see [2].

Any $(d+2)$-element point set in general position in $\mathbb{R}^d$ has a unique *Radon partition*, that is, a partition into two parts, X and Y, such that $\operatorname{conv} X$ and $\operatorname{conv} Y$ have precisely one point in common. For a survey on Radon's theorem, see [5].

We start with some elementary observations concerning $t(A)$, the *maximum* number of simplices in a triangulation of A.

Claim 4. *Let $A \subset \mathbb{R}^d$ be a finite set of points in general position with $|A| \geq d+2$. Let a_0 and a_1 denote the number of vertices of* $\operatorname{conv} A$ *and the number of its interior points, respectively, so that $a_0 + a_1 = |A|$. Then, for the maximum number of simplices in a triangulation of A, we have*

1. *$t(A) \geq a_0 + da_1 - d$,*
2. *$t(A) \geq |A| - \lfloor \frac{d+2}{2} \rfloor$.*

The first bound is always at least as strong as the second one, unless $a_1 = 0$ and $d \geq 3$.

Proof. 1. It is easily seen, by induction on d, that there exists a triangulation $\mathcal{T}$ of the vertex set of $\operatorname{conv} A$, using at least $a_0 - d$ simplices [9]. Any point $p \in A$ that lies in the interior of $\operatorname{conv} A$ belongs to a unique simplex of $S \in \mathcal{T}$. Subdividing S from p into $d+1$ smaller simplices, we increase the number of simplices by d. Including successively all other interior points, we obtain a triangulation of A, consisting of at least $a_0 - d + a_1 d$ simplices.

2. Fix any $(d+2)$-element subset $A' \subset A$, and consider its (unique) Radon partition $A' = X \cup Y$, where $|X| \geq |Y|$, so that we have $|X| = m \geq \lceil \frac{d+2}{2} \rceil$. Notice

that the simplices $S_x = \text{conv}\, A' \setminus \{x\}$, for all $x \in X$, form a triangulation of A'. (This fact is essentially equivalent to a theorem of Proskuryakov [8], which states that any two points of X are separated by the hyperplane spanned by the remaining points of A'.) Thus, A' can be triangulated using m simplices. Every additional point of A can be included in this triangulation, increasing the number of simplices by at least one. This yields that

$$t(A) \geq m + |A \setminus A'| \geq \left\lceil \frac{d+2}{2} \right\rceil + |A| - (d+2) \geq |A| - \left\lfloor \frac{d+2}{2} \right\rfloor,$$

as desired. □

Now we are in a position to establish Theorem 1. Let $d \geq 2$. Suppose for contradiction that there is a set P of more than $d+2$ points in general position in $\mathbb{R}^{d+1}$ that surrounds the origin 0. Fix a d-dimensional hyperplane Π that does not pass through 0 and is not parallel to any line $0p$ $(p \in P)$. For any $p \in P$, either the ray $\overrightarrow{0p}$ or its reflection about the origin, $-\overrightarrow{0p}$, intersects Π. In the former case, assign to p the intersection point $a_p = \overrightarrow{0p} \cap \Pi$, and color it *amber*. In the latter one, assign to p the point $b_p = (-\overrightarrow{0p}) \cap \Pi$, and color it *blue*. Let A and B denote the sets of amber and blue points, and let $S = A \cup B$. Clearly, $A \cap B = \emptyset$ and the set S is in general position in $\mathbb{R}^d$.

The assumption that P surrounds the origin implies (in fact, is equivalent to the condition) that the two-colored set $S \subset \mathbb{R}^d$ satisfies the following property.

Claim 5. *The set $S = A \cup B$ $(A, B \neq \emptyset)$ is in general position in $\mathbb{R}^d$ and $|S| > d+2$. For every $T \subset S$, $|T| = d+1$, there exists $x \in S \setminus T$ such that*

$$\text{conv}\,(T \cup \{x\}) \cap A \cap \text{conv}\,(T \cup \{x\}) \cap B \neq \emptyset.$$

In particular, every full-dimensional simplex whose vertices belong to A contains at least one element of B, and every full-dimensional simplex whose vertices belong to B contains at least one element of A.

Notice that the above construction can be carried out for any d-dimensional hyperplane Π. Changing Π will result in a *projective transformation* of the point set $S = S(\Pi)$. In the process, the color of a point p changes as it passes through a position "at infinity" (as Π becomes parallel to the line $0p$).

Choose a projective transformation that maximizes $|A| - |B|$, that is, for which the number of amber points is as large as possible. In the sequel, A and B stands for the corresponding sets of amber and blue points.

Claim 6. *The sets A and B satisfy*

1. $|H \cap A| \geq |H \cap B|$*, for any half-space* H*;*

2. $B \subset \operatorname{conv} A$*;*

3. $|A| \geq d+2$*;*

Proof. 1. If in some open half-space H, the points from B outnumbered the points from A, then a projective transformation that takes the supporting hyperplane of H to infinity would increase $|A|-|B|$.

2. Choosing Π to be a d-flat induced by any $d+1$ elements of P, the resulting d-dimensional set has at least $d+1$ amber points. Therefore, in the optimal configuration, $A \subset \mathbb{R}^d$ is full-dimensional and $|A| \geq d+1$. Applying part 1 of the statement to the half-spaces bounded by the facets of $\operatorname{conv} A$ and disjoint from the interior of $\operatorname{conv} A$, we obtain that no point of B can lie outside of the convex hull of A.

3. Suppose for contradiction that $|A| = d+1$. Since $|S| > d+2$, there are at least two elements of B in the interior of $\operatorname{conv} A$. Then the interior of any supporting half-space of the simplex $\operatorname{conv} A$, whose boundary hyperplane passes through a facet of $\operatorname{conv} A$, contains two blue points and only one amber point. This contradicts part 1. □

We distinguish two cases.

Case 1: $a_1 > 0$*, i.e., the interior of* $\operatorname{conv} A$ *contains at least one point of* A*.*

Let $a_0 = d+1+x$ and $a_1 = 1+y$, for some non-negative numbers x and y. By Claim 5, S does not contain empty monochromatic full-dimensional simplices. In particular, every amber simplex contains at least one blue point in its interior, so that

$$|B| \geq t(A) \geq 1+x+d(1+y),$$

where the second inequality follows by Claim 4(1). Applying the same claim to B, we obtain that

$$t(B) \geq |B|-d \geq 1+x+d(1+y)-d = 1+x+dy.$$

On the other hand, every blue simplex contains at least one amber point. By Claim 6(2), such a point must belong to the *interior* of $\operatorname{conv} A$. Therefore, we have

$$a_1 = 1+y \geq t(B).$$

Comparing the last two inequalities, we obtain that $x = y = 0$ and $|B| \geq d+1$.

The case $a_0 = d+1$, $a_1 = 1$, $|B| \geq d+1$ is impossible: On one of the sides of a hyperplane containing a facet of $\operatorname{conv} B$ there are fewer than d points of A. By Claim 6(1), this violates the maximality of $|A|-|B|$.

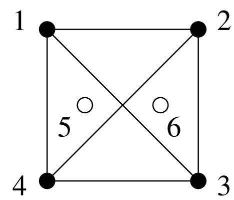

Figure 1: $\{x,2,3,5\}$ does not induce a Radon-partition for any $x \in \{1,4,6\}$.

Case 2: $a_1 = 0$*, i.e., A is in convex position.*

Now we have $|B| < d+1$, otherwise there is a full-dimensional empty blue simplex, contradicting Claim 5. Therefore, B is contained in some hyperplane h. Applying part 1 of Claim 6 to a half-space bounded by h that contains at most as many elements of A as the other, we obtain that

$$|B| \leq \left\lfloor \frac{|A|}{2} \right\rfloor.$$

On the other hand, as in Case 1, we have

$$|B| \geq t(A) \geq |A| - \left\lfloor \frac{d+2}{2} \right\rfloor.$$

Comparing the upper and lower bounds on $|B|$ and taking Claim 6(3) into account, we obtain that d is even and $|A| = d+2$.

If $d > 2$, then $|B| < d$, so B is contained in an infinite family of hyperplanes that sweep through $\mathbb{R}^d$. One of these hyperplanes must pass through a point of A, which gives us a half-space that contains more points from B than from A, contradicting the maximality of $|A| - |B|$.

The remaining case is $d = 2$: The 4 points of A are in convex position, and the 2 points of B lie in opposite quadrants spanned by the diagonals of conv A. It is easily seen that this configuration violates Claim 5. See Fig. 1 (For $T = \{2,3,5\}$, there is no suitable x). This completes the proof of Theorem 1.

3 k-surrounding sets

A point set P in general position with respect to the origin *0* is said to be *k-surrounding*, or is said to have *property $S(k)$,* if any k-element subset of P can be extended to a $(d+1)$-element subset that contains *0* in its interior.

Proof of Theorem 3. The case when d is odd follows from the case when d is even. To see this, suppose $P \subset \mathbb{R}^{2n}$ has property $S(n+1)$, and consider P as a subset of the hyperplane $\{(x_1,\ldots,x_{2n},-1)\} \subset \mathbb{R}^{2n+1}$, such that P surrounds the point $(0,\ldots,0,-1)$. Let $Q = P \cup \{(0,\ldots,0,1)\}$. It is easily seen that Q has property $S(n+1)$: Let $X \subset Q$ be of size $n+1$. If $X \subset P$, there exists a set $Y \subset P$ with $|Y| = n$ such that $(0,\ldots,0,-1) \in \text{conv } X \cup X'$. Then the origin is contained in $\text{conv } X \cup X' \cup \{(0,\ldots,0,1)\}$. Otherwise, $X = X' \cup \{(0,\ldots,0,1)\}$, where $X' \subset P$ and $|X'| = n$. Taking into account that property $S(n+1)$ implies property $S(n)$, there exists a set $Y \subset P$ with $|Y| = n+1$ such that $(0,\ldots,0,-1) \in \text{conv } X' \cup Y$, and consequently, the origin is contained in $\text{conv } X \cup Y$. Therefore, it suffices to consider the case when d is even.

To complete the proof of Theorem 3, it will be more convenient to transform the problem via the well known Gale transform. (For details concerning the Gale transform, we refer the reader to [6] or [12].)

Let $d \geq 2$ be an integer and suppose $P \subset \mathbb{R}^d$ is in general position with respect to the origin, $|P| = n$, and P has property $S(k)$. The Gale transform of $P \cup \{0\}$ is a $(|P|+1)$-element vector configuration in $\mathbb{R}^{n-d}$, which we denote by $\mathbf{V} \cup \{\mathbf{1}\}$. Here $|\mathbf{V}| = n$ and the vector $\mathbf{1}$ corresponds to the origin *0* in the "primal" space.

Property $S(k)$ corresponds to the following property of $\mathbf{V}$: For every $\mathbf{U} \subset \mathbf{V}$ with $|\mathbf{U}| = n-k$, there exists $\mathbf{W} \subset \mathbf{U}$ with $|\mathbf{W}| = n-d-1$, such that $(\mathbf{n_W} \cdot \mathbf{1})(\mathbf{n_W} \cdot \mathbf{v}) < 0$ for every $\mathbf{v} \in \mathbf{V} \setminus \mathbf{W}$. Here, $\mathbf{n_W}$, is some fixed vector orthogonal to $\mathbf{W}$, and, $\cdot$, denotes the usual dot product.

In particular, property $S(k)$ implies that there is an $(n-d-1)$-dimensional hyperplane H through the origin with normal vector $\mathbf{n}$ such that $(\mathbf{n}\cdot\mathbf{1})(\mathbf{n}\cdot\mathbf{v}) < 0$ for every $\mathbf{v} \in \mathbf{V}$. Therefore, if we extend the vectors of $\mathbf{V}$ to rays, they will intersect $H - \mathbf{1}$. The set of intersection points, P^*, is a set of n points in general position in $\mathbb{R}^{n-d-1}$ with the following property, denoted by $S^*(k)$: *Among any $n-k$ points of P^*, there are some $n-d-1$ that form a facet of* $\text{conv } P^*$.

In fact, this necessary condition is also sufficient, for one can choose an appropriate vector $\mathbf{1}$ in $\mathbb{R}^{n-d}$ which corresponds to a point set $P \subset \mathbb{R}^d$ with property $S(k)$. Summarizing:

Observation 7. *There exist n points in $\mathbb{R}^d$ satisfying property $S(k)$ if and only if there exist n points in $\mathbb{R}^{n-d-1}$ satisfying property $S^*(k)$.*

We now complete the proof of Theorem 3. First note that for $d = 2$, the regular $2n+1$-gon has property $S(2)$. It remains to exhibit arbitrarily large sets point sets in $\mathbb{R}^d$ for even $d \geq 4$ with property $S(d/2+1)$.

For positive integers k and $n > 2k-1$, let $C(n,k)$ denote the cyclic polytope on n vertices in $\mathbb{R}^{n-2k+1}$. The facets of $C(n,k)$ have a simple characterization known as Gale's evenness condition (see [6] or [12]). Using this characterization, it is easy to show that when n is odd, $C(n,k)$ has property $S^*(k)$. Hence, by Observation 7, there exist n points in $\mathbb{R}^{2k-2}$ with property $S(k)$. $\square$

Remark. For the particular point sets we get from $C(n,k)$ it is easily checked, using Gale's evenness condition, that the maximum t for which there exists colorings of the points with t colors and no rainbow simplex containing the origin, is $\frac{n+2k-1}{2}$.

By Observation 7, Problem 2 can be reformulated in terms of the property $S^*(k)$. We obtain the following.

Problem 8. *Let $d \geq 2$ be fixed.*

1. *What is the largest integer $k = k(d)$ such that there exists arbitrarily large finite point sets P in general position in $\mathbb{R}^{|P|-d-1}$ that have property $S^*(k)$?*
2. *What is the smallest integer $K = K(d)$ such that there exists no finite point set P in general position in $\mathbb{R}^{|P|-d-1}$ with more than $d+1$ elements, which has property $S^*(K+1)$?*

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