ARRANGEMENTS OF HOMOTHETS OF A CONVEX BODY

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Abstract. Answering a question of Füredi and Loeb (1994) we show that the maximum number of pairwise intersecting homothets of a d-dimensional centrally symmetric convex body, each not containing the others' centers in their interiors is at most $O(3^d d \log d)$. We also show the bound of $O(3^d \binom{2d}{d} d \log d)$ for arbitrary convex bodies each not containing the others' centroids, as well as a generalization where the center is an arbitrary point chosen from the interior. We present exponential lower bounds in all cases.

1. Introduction

A *convex body* K in the Euclidean d-dimensional space $\mathbb{R}^d$ is a compact convex set with non-empty interior. A (positive) *homothet* of K is a set of the form $\lambda K + v := \{\lambda k + v \ : \ k \in K\}$, where $\lambda > 0$ is the homothety ratio, and $v \in \mathbb{R}^d$ is a translation vector. We investigate arrangements of homothets of convex bodies. The starting point of our investigations is Problem 4.4 of a paper of Füredi and Loeb [FL94]:

> *Is it true that for any centrally symmetric body K of dimension $d, d \geq d_0$, the number of pairwise intersecting homothetic copies of K which do not contain each other's centers is at most 2^d?*

They observe that for the Euclidean plane, there exist 8 such homothets of the disc [MM92, HJLM93] (see Fig. 1).

Definition 1. Let K be a convex body and p a point in $\mathbb{R}^d$. We extend a notion of L. Fejes Tóth by defining a *Minkowski arrangement of K with respect to p* to be a family $\{v_i + \lambda_i K\}$ (with $\lambda_i > 0$ for all i) of homothets of K with the property that $v_i + p$ is not in $v_j + \lambda_j \operatorname{int}(K)$, for any $i \neq j$. We denote the largest number of homothets that a pairwise intersecting Minkowski arrangement of K with respect to p can have by $\kappa(K, p)$.

Similarly, we define a *strict Minkowski arrangement of K with respect to p* to be a family $\{v_i + \lambda_i K\}$ of positive homothets of K such that $v_i + p \notin v_j + \lambda_j K$, for any $i \neq j$.

We denote the largest number of homothets that a pairwise intersecting strict Minkowski arrangement of K with respect to p can have by $\kappa'(K, p)$. When K is symmetric about the origin and $p = o$ is the origin, we omit p and write $\kappa(K)$ and $\kappa'(K)$.

Thus, the question of Füredi and Loeb may be phrased as: *Is it true that $\kappa'(K) \leq 2^d$ for any o-symmetric convex body K in $\mathbb{R}^d$ with $d \geq d_0$ for some constant d_0?*

A negative answer is simply seen as follows. Suppose that we are given a collection $\{v_1, \ldots, v_m\} \subset \mathbb{R}^d$ of points such that $\|v_i\|_K = 1$ for all i and $\|v_i - v_j\|_K > 1$ for all distinct i, j. The largest such m is known as the strict Hadwiger number of K, denoted $H'(K)$. Then $\{K + v_i : i = 1, \ldots, m\}$ is a strict Minkowski arrangement of translates of K all intersecting in o, hence $\kappa'(K) \geq H'(K)$. Thus, it is sufficient to find an o-symmetric convex body K with $H'(K) > 2^d$. In dimension 3 we may take the Euclidean ball B^3, for which it is well known that $H'(B^3) = 12$. For $d > 3$ we may use a result of Talata [Tal05, Lemma 3.1] that asserts that $H'(C^k \times K) = 2^k H'(K)$

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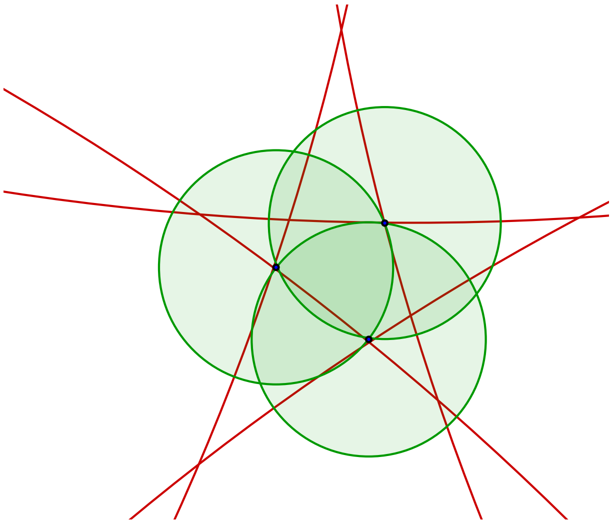

FIGURE 1. A pairwise intersecting strict Minkowski arrangement of 8 circles (after Harary et al. [HJLM93])

for any o-symmetric convex body K, where C^k is the k-dimensional cube. In particular, $H'(B^3 \times C^{d-3}) = 3 \cdot 2^{d-1}$ for all $d \geq 3$. In fact, Talata [Tal05] constructs d-dimensional o-symmetric convex bodies K such that $H'(K) \geq \frac{16}{35}\sqrt{7}^d$ for all $d \geq 3$. The question now becomes: how large can $\kappa'(K)$ be? One goal of this paper is to present bounds on κ and κ'.

We recall the definition of some related quantities.

Definition 2. The *Hadwiger number* (resp., *strict Hadwiger number*) of K is defined as the maximum number $H(K)$ (resp., $H'(K)$) of non-overlapping (resp., disjoint) translates of K touching K. When K is o-symmetric, $H(K)$ equals the maximum number of points $v_1, \dots, v_m$ such that $\|v_i\|_K = 1$ for all i and $\|v_i - v_j\|_K \geq 1$ for all distinct i, j. In this case, $\{K\} \cup \{K + v_i : i = 1, \dots, m\}$ is a Minkowski arrangement of translates of K all intersecting in o, hence $\kappa(K) \geq H(K) + 1$.

If K is o-symmetric, we define the *packing number* $P(K, \lambda)$ of K as the maximum number of points in the normed space with unit ball K, such that the ratio of the maximal distance to the minimal distance is at most λ. We denote the normed space with unit ball K as $\mathcal{N}$, and use the notations $\kappa(\mathcal{N}), P(\mathcal{N}, \lambda), H(\mathcal{N}), \dots$ in place of $\kappa(K), P(K, \lambda), H(K), \dots$.

It follows from the isodiametric inequality in normed spaces (an immediate corollary to the Brunn-Minkowski Theorem [Busemann 1947, Mel'nikov 1963]) that

$$P(\mathcal{N}, \lambda) \leq (1 + \lambda)^d \tag{1}$$

for any d-dimensional normed space $\mathcal{N}$. (See Lemma 6 below for a generalization.) Our first result is an exponential upper bound on κ in the case when K is o-symmetric.

Theorem 3. *Let $\mathcal{N}$ be a d-dimensional real normed space. Then*

$$\kappa'(\mathcal{N}) \leq \kappa(\mathcal{N}) \leq P\left(\mathcal{N}, 2(1 + \tfrac{1}{d})\right)(d + O(1)) \log d = O(3^d d \log d).$$

Note that $\kappa(C^d) \geq H(C^d) + 1 = 3^d$, which shows that Theorem 3 is sharp up to the $O(d \log d)$ factor. Theorem 3 is a special case of Theorem 8 below that also deals with non-symmetric K. Next consider any convex body K (not necessarily o-symmetric). It is easy to see that $\kappa(K, p)$ is infinite if p is not in the interior of K. Moreover, $\kappa'(K, p)$ is infinite if $p \notin K$ or, slightly more generally, if there is a line ℓ through p such that $K \cap \ell \subseteq \{p\}$. We therefore restrict p to be in the interior of K.

Definition 4. Let K be a convex body with p in its interior. Define $\theta(K,p)$, the *measure of asymmetry of K with respect to p* to be $\theta(K,p) := \sup\{\theta\colon p-K \subseteq \theta(K-p)\}$. (For a similar looking quantity, see [Grü63, Section 6.1].) If K contains the origin in the interior, we define the (asymmetric) norm $\|\cdot\|_K : \mathbb{R}^d \to \mathbb{R}$ by $\|x\|_K = \inf\{\lambda > 0\colon x \in \lambda K\}$.

Note that $\theta(K,o) = \sup\{\|x\|_K / \|-x\|_K : x \in \operatorname{bd} K\}$. We will also use the (symmetric) norm defined by the unit ball $K \cap -K$. Thus, $\|x\|_{K\cap -K} = \max\{\|x\|_K, \|-x\|_K\}$. We also need a third symmetric norm.

Definition 5. For any convex body K, define its *central symmetral* to be $\frac{1}{2}(K-K)$. If $o \in \operatorname{int}(K)$, then $P(K,\lambda)$ is defined to be the maximum number of points $p_1,\dots,p_m$, such that $\|p_i - p_j\|_{\frac{1}{2}(K-K)} / \|p_i - p_j\|_K \le \lambda$ for all distinct $i,j = 1,\dots,m$.

If K is o-symmetric, then the norms $\|\cdot\|_K$, $\|\cdot\|_{K\cap -K}$, and $\|\cdot\|_{K\cap -K}$ are all identical, and $P(K,\lambda)$ coincides with the definition given before.

Lemma 6. *For any convex body K with o in its interior and any $\lambda > 0$,*

$$P(K,\lambda) \le (\lambda+1)^d \frac{\operatorname{vol}(\frac{1}{2}(K-K))}{\operatorname{vol}(K\cap -K)}.$$

We also need to generalize the Hadwiger number to the non-symmetric case, in the following non-standard way.

Definition 7. If $o \in \operatorname{int}(K)$, define $h(K)$ to be the maximum number of points $p_1,\dots,p_m$ on $\operatorname{bd} K$ such that $\|p_i - p_j\|_K \ge 1$ for all distinct $i,j = 1,\dots,m$. Similarly, we define $h'(K)$ to be the maximum number of points $p_1,\dots,p_m \in \operatorname{bd} K$ such that $\|p_i - p_j\|_K > 1$ for all distinct $i,j = 1,\dots,m$.

Note that if $K = -K$, then $h(K) = H(K)$ and $h'(K) = H'(K)$ (cf. Definition 2). This is not necessarily the case if K is not o-symmetric. Generalizing our observation for the symmetric case above, if $p_1,\dots,p_m \in \operatorname{bd} K$ satisfy $\|p_i - p_j\|_K > 1$ for all distinct i,j, then the collection $\{K - p_i\colon i = 1,\dots,m\}$ is a pairwise intersecting strict Minkowski arrangement of translates of K, hence $\kappa'(K,o) \ge h'(K)$. Similarly (by adding K to the collection) we have $\kappa(K,o) \ge h(K)+1$.

Theorem 8. *Let K be a convex body in $\mathbb{R}^d$ with $o \in \operatorname{int}(K)$. Then*

$$\kappa'(K,o) \le \kappa(K,o) \le P\left(K, 2(1+\tfrac{1}{d})\right)(d + O(1)) \log\theta(K,o)d.$$

From this theorem and some other well-known results we can easily deduce the following estimates.

Corollary 9. *Let K be a convex body in $\mathbb{R}^d$ with $p \in \operatorname{int}(K)$. Then*

$$\kappa'(K,p) \le \kappa(K,p) \le \left(\frac{3}{2}\right)^d \frac{\operatorname{vol}(K-K)}{\operatorname{vol}((K-p)\cap(p-K))} O(d\log\theta(K,p)d).$$

If c is the centroid of K then

$$\kappa(K,c) \le P\left(K, 2(1+\tfrac{1}{d})\right)(2d+O(1))\log d \le 3^d \binom{2d}{d} O(d\log d).$$

The following is an example of a d-dimensional convex body K for which $\kappa(K,c) \ggg 3^d = \kappa(C^d)$. Note that $\kappa(\Delta,o) = 10$, where Δ is a triangle with centroid o [FT95] (see Fig. 2). A Cartesian product of $d/2$ triangles gives a d-dimensional convex body C with centroid o such that $\kappa(C,o) \ge \sqrt{10}^d$.

We prove Theorem 8 and Corollary 9 in Section 2.

When K is o-symmetric, there is a lower bound $\Omega((2/\sqrt{3})^d)$ on $H'(K)$ [AdRBV98, Theorem 1], which implies that $\kappa'(K) = \Omega((2/\sqrt{3})^d)$. Before the result in [AdRBV98], Bourgain

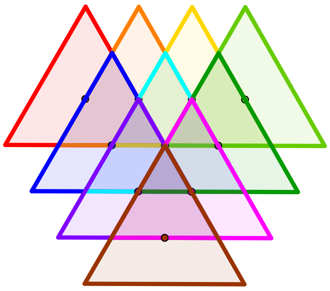

FIGURE 2. A pairwise intersecting Minkowski arrangement of 10 triangles [FT95]

[FL94] showed an exponential lower bound to $H'(K)$ that depends only on the dimension of K. (This argument was also independently discovered by Talata [Tal98].) The key tool used by Bourgain and Talata is the Quotient of Subspace Theorem (or, in short, the QS Theorem) of Milman [Mil85], which states the following.

Let $1 \le k < d$, and $\lambda = k/d$. Let K be a convex body in $\mathbb{R}^d$. Then there is a projection P of $\mathbb{R}^d$ onto a subspace F and a subspace E of F, and an ellipsoid $\mathcal{E}$ in E such that $\dim E = k$ and

$$\mathcal{E} \subseteq P(K) \cap E \subseteq c(\lambda)\mathcal{E},$$

where $c(\lambda)$ depends only on λ.

In order to obtain a lower bound on $\kappa(K,p)$ in the non-symmetric case, valid for all reference points $p \in \mathbb{R}^d$, the QS Theorem has to be extended to non-symmetric convex bodies. Such a non-symmetric QS Theorem can be found in Milman and Pajor [MP00]. However, the centroid of K plays a special role in their result, as E and F have to be affine subspaces through the centroid. To bypass this limitation, we prove the following topological result.

Lemma 10 ("Centroid of Projection Lemma"). *Let K be a convex body in $\mathbb{R}^d$. Then there is a $(d-1)$-dimensional linear subspace H of $\mathbb{R}^d$ such that the centroid of the orthogonal projection of K onto H is the origin.*

Statements similar to this lemma are known (see for instance [Izm14]), and most likely, so is the lemma itself. However, we did not find a reference where this result is explicitly stated or where it clearly follows from stated results. In Section 3, we present a proof of Lemma 10, and show how the following theorem follows from this lemma and the above quoted result of Milman and Pajor.

Theorem 11. *Let K be a convex body and p a point in $\mathbb{R}^d$. Then $c^d < \kappa'(K,p) \le \kappa(K,p)$ for some universal constant $c > 1$.*

2. BOUNDING κ FROM ABOVE

Proof of Lemma 6. Let $T \subset \mathbb{R}^d$ be such that $\|x-y\|_{K\cap -K} \ge 1$ for all distinct $x, y \in T$ and $\|x-y\|_{\frac{1}{2}(K-K)} \le \lambda$. Then $\{v + \frac{1}{2}(K \cap -K) : v \in T\}$ is a packing. Let $P = T + \frac{1}{2}(K \cap -K)$. Then $\operatorname{vol}(P) = 2^{-d}|T| \operatorname{vol}(K \cap -K)$ and

$$P - P = T - T + (K \cap -K) \subset \frac{\lambda}{2}(K-K) + \frac{1}{2}(K-K) = \frac{\lambda+1}{2}(K-K).$$

By the Brunn-Minkowski inequality, $\operatorname{vol}(P-P) \ge 2^d \operatorname{vol}(P)$, and it follows that

$$|T| = \frac{2^d \operatorname{vol}(P)}{\operatorname{vol}(K \cap -K)} \le \frac{\operatorname{vol}(P-P)}{\operatorname{vol}(K \cap -K)} \le \frac{(\lambda+1)^d \operatorname{vol}(\frac{1}{2}(K-K))}{\operatorname{vol}(K \cap -K)}. \qquad \square$$

Before we prove Theorem 8, we first show an extension of the so-called "bow-and-arrow" inequality of [FL94] (Corollary 14) to the case of an asymmetric norm.

Definition 12. For any non-zero $v \in \mathbb{R}^d$ write $\widehat{v} = \frac{1}{\|v\|_K} v$ for the normalization of v with respect to $\|\cdot\|_K$.

We will only consider normalizations with respect to $\|\cdot\|_K$, and never with respect to $\|\cdot\|_{K\cap -K}$ or $\|\cdot\|_{\frac{1}{2}(K-K)}$.

Lemma 13. *Let K be a convex body in $\mathbb{R}^d$ containing o in its interior. Let $a, b \in \mathbb{R}^d$ such that $\|a\|_K \geq \|b\|_K > 0$. Then*

$$\left\|\widehat{a} - \widehat{b}\right\|_K \geq \frac{\|a-b\|_K - \|a\|_K + \|b\|_K}{\|b\|_K}.$$

Proof.

$$\begin{aligned}\|a-b\|_K &= \left\|\|a\|_K \widehat{a} - \|b\|_K \widehat{b}\right\|_K \\ &= \left\|\|b\|_K (\widehat{a} - \widehat{b}) + (\|a\|_K - \|b\|_K)\widehat{a}\right\|_K \\ &\leq \|b\|_K \left\|\widehat{a} - \widehat{b}\right\|_K + \|a\|_K - \|b\|_K . \end{aligned}$$ □

Corollary 14. *For any two non-zero elements a and b of a normed space,*

$$\left\|\widehat{a} - \widehat{b}\right\| \geq \frac{\|a-b\| - |\|a\| - \|b\||}{\|b\|}.$$

Proof of Theorem 8. Let the pairwise intersecting Minkowski arrangement be $\{\lambda_i K + v_i : i = 1, \ldots, m\}$. Without loss of generality, $\lambda_1 = \min_i \lambda_i = 1$ and $v_1 = o$. Given $N \in \mathbb{N}$ and $\delta > 0$, we partition the Minkowski arrangement into N subarrangements as follows. Let $I_j = \{i : \lambda_i \in [(1+\delta)^{j-1}, (1+\delta)^j]\}$ for each $j = 1, \ldots, N$, and let $I_\infty = \{i : \lambda_i \in [(1+\delta)^N, \infty)\}$. We bound the size of each subarrangement $\{\lambda_i K + v_i : i \in I_j\}$, $j \in \{1, \ldots, N, \infty\}$, separately. Finally, we choose appropriate values for N and δ.

The next lemma bounds I_j, $j \neq \infty$, in terms of δ and K.

Lemma 15. *Let K be a d-dimensional convex body with $o \in \operatorname{int}(K)$. Let $\{v_i + \lambda_i K : i \in I\}$ be a pairwise intersecting Minkowski arrangement of positive homothets of K, with $\lambda_i \in [1, 1+\delta)$ for each $i \in I$. Then*

$$|I| \leq P\left(K, 2(1+\delta)\right).$$

Proof. Write $T = \{v_i : i \in I\}$. Since any two homothets intersect, $\|v_i - v_j\|_{\frac{1}{2}(K-K)} \leq 2(1+\delta)$. Since $v_i \notin v_j + \lambda_j \operatorname{int}(K)$, it follows that $v_i - v_j \notin \operatorname{int}(K \cap -K)$ for all distinct $i, j \in I$, which gives that $\|v_i - v_j\|_{K\cap -K} \geq 1$. □

The following lemma is used to bound I_∞.

Lemma 16. *Let K be a d-dimensional convex body with $o \in \operatorname{int}(K)$. Let $\{v_i + \lambda_i K : i \in I\}$ be a Minkowski arrangement of positive homothets of K with $\lambda_i \geq 1$, $(v_i + \lambda_i K) \cap -\varepsilon K \neq \emptyset$ and $o \notin v_i + \lambda_i \operatorname{int}(K)$ for all $i \in I$. Then*

$$|I| \leq P\left(K, \frac{2}{1-\varepsilon}\right).$$

We first consider two homothets in the Minkowski arrangement of the previous lemma.

Lemma 17. *Let $v_1 + \lambda_1 K$ and $v_2 + \lambda_2 K$ be two positive homothets of K such that $\lambda_1, \lambda_2 \geq 1$, $v_1 \notin v_2 + \lambda_2 \operatorname{int}(K)$, $v_2 \notin v_1 + \lambda_1 \operatorname{int}(K)$, $o \notin v_i + \lambda_i \operatorname{int}(K)$ and $(v_i + \lambda_i K) \cap -\varepsilon K \neq \emptyset$ $(i = 1, 2)$. Then $\left\|\frac{1}{\|-v_1\|_K}(-v_1) - \frac{1}{\|-v_2\|_K}(-v_2)\right\|_{K\cap -K} \geq 1 - \varepsilon$.*

Proof. Since $\|\cdot\|_{K\cap -K}$ is symmetric, we may assume that $\|-v_1\|_K \leq \|-v_2\|_K$. Since $(v_1+\lambda_1 K)\cap -\varepsilon K \neq \emptyset$, $v_1 + \lambda_1 x = -\varepsilon y$ for some $x, y \in K$. Therefore, $\|-v_1\|_K \leq \lambda_1 \|x\|_K + \varepsilon \|y\|_K \leq \lambda_1 + \varepsilon$. Also, since $o \notin v_1 + \lambda_1 \operatorname{int}(K)$, we have that $\|-v_1\|_K \geq \lambda_1$. Similarly, $\lambda_2 \leq \|-v_2\|_K \leq \lambda_2 + \varepsilon$. Since $v_1 \notin v_2 + \lambda_2 \operatorname{int}(K)$, we obtain that $\|v_1 - v_2\|_K \geq \lambda_2$. We apply Lemma 13 to obtain

$$\begin{aligned}
\left\|\widehat{-v_1} - \widehat{-v_2}\right\|_{K\cap -K} &\geq \left\|\widehat{-v_2} - \widehat{-v_1}\right\|_K \\
&\geq \frac{\|v_1 - v_2\|_K - \|-v_2\|_K + \|-v_1\|_K}{\|-v_1\|_K} \\
&\geq \frac{\lambda_2 - (\lambda_2 + \varepsilon) + \|-v_1\|_K}{\|-v_1\|_K} \\
&= 1 - \frac{\varepsilon}{\|-v_1\|_K} \geq 1 - \frac{\varepsilon}{\lambda_1} \geq 1 - \varepsilon. \qquad \square
\end{aligned}$$

Proof of Lemma 16. For each $i \in I$, let $t_i = \widehat{-v_i}$. Let $T := \{t_i \colon i \in I\}$. By Lemma 17, $\|t_i - t_j\|_{K\cap -K} \geq 1 - \varepsilon$ for all distinct $i, j \in I$. Since $T \subset K$, $\|t_i - t_j\|_{\frac{1}{2}(K-K)} \leq 2$. It follows that $|I| \leq P\left(K, 2/(1-\varepsilon)\right)$. $\square$

We now finish the proof of Theorem 8. By Lemma 15, for $j = 1, \ldots, N$, $|I_j| \leq P\left(K, 2(1+\delta)\right)$, and by Lemma 16 applied to I_∞ and $\varepsilon = \theta(K, o)(1+\delta)^{-N}$,

$$|I_\infty| \leq P\left(K, \frac{2}{1 - \theta(K, o)(1+\delta)^{-N}}\right).$$

It follows that

$$m = N\sum_{j=1}^{N} |I_j| + |I_\infty| \leq P\left(K, 2(1+\delta)\right) + P\left(K, \frac{2}{1 - \theta(K, o)(1+\delta)^{-N}}\right).$$

We now choose

$$N := 1 + \left\lceil \frac{\log \theta(K, o) d}{\log(1 + \frac{1}{d})} \right\rceil = (d + O(1)) O(\log \theta(K, o) d)$$

and $\delta = 1/d$. Then

$$N \geq 1 + \frac{\log \theta(K, o) d}{\log(1 + \frac{1}{d})},$$

which implies that

$$\frac{2}{1 - \theta(K, o)(1+\delta)^{-N}} \leq 2(1+\delta),$$

hence

$$m \leq P\left(K, 2(1 + \tfrac{1}{d})\right)(N+1) = P\left(K, 2(1 + \tfrac{1}{d})\right)(d + O(1)) \log \theta(K, o) d. \qquad \square$$

Proof of Corollary 9. The first statement follows from Theorem 8 combined with Lemma 6.

If o is the centroid of K, then it is well known (the earliest appearance of this fact may be in [Min97]) that $\theta(K, o) \leq d$. Also, by a result of Milman and Pajor [MP00, Corollary 3] for a convex body K with centroid o, $\operatorname{vol}(K)/\operatorname{vol}(K \cap -K) \leq 2^d$, which, together with the Rogers-Shephard inequality [RS57] $\operatorname{vol}(K-K) \leq \binom{2d}{d} \operatorname{vol}(K)$, gives the second statement. $\square$

3. Bounding κ' from below

Proof of Lemma 10. For any unit vector $u \in \mathbb{S}^{d-1}$, let $f(u)$ be the centroid of the orthogonal projection of K onto $u^{\perp}$. We need to show that $f(u) = o$ for some $u \in \mathbb{S}^{d-1}$. Suppose not. Then $\widehat{f}\colon \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ defined by $\widehat{f}(u) = f(u)/\left\|f(u)\right\|_2$ is a continuous, even mapping such that $\langle u, f(u)\rangle = 0$ for all $u \in \mathbb{S}^{d-1}$. Since f is even, its degree is even (see for instance [Hat02, Proposition 2.30]. Also, $f(u) \neq -u$ for all $u \in \mathbb{S}^{d-1}$. It follows that f is homotopic to the identity map, which has degree 1, a contradiction. □

The non-symmetric version of the QS theorem, due to Milman and Pajor [MP00, Theorem 9], combined with Lemma 10 yields the following.

Theorem 18. *Let $1 \leq k < d-1$, and $\lambda = k/(d-1)$. Let K be a convex body in $\mathbb{R}^d$. Then there is a projection P of $\mathbb{R}^d$ onto a subspace F and a subspace E of F, and an ellipsoid $\mathcal{E}$ in E such that* $\dim E = k$ *and*

$$\mathcal{E} \subseteq P(K) \cap E \subseteq c(\lambda)\mathcal{E},$$

where $c(\lambda)$ depends only on λ.

Finally, the same proof as the one that yields Theorem 4.3 in [FL94], now yields Theorem 11.

Proof of Theorem 11. We closely follow the proof of the symmetric case (Theorem 4.3) in [FL94].

By Theorem 18, there is a roughly $(d/2)$-dimensional subspace E, such that for an appropriate projection P of $\mathbb{R}^d$, we have $\mathcal{E} \subseteq P(K) \cap E \subseteq c\mathcal{E}$ with some universal constant c. By a theorem of Milman [Mil71] (see also [MS86, Section 4.3]), we can take a $C(d/2)$-dimensional subspace E' of E such that $\mathcal{E} \subseteq P(K) \cap E \subseteq 1.1\mathcal{E}$, for a universal constant $C > 0$. Although this is stated only for symmetric bodies K in [MS86], the proof obviously works in the non-symmetric case as well. Now, there are exponentially many points on the relative boundary of $K' := P(K) \cap E'$ such that the distance (with respect to the non-symmetric norm on E' whose unit ball is K') between any two points is at least 1.21. Let X be the set of these points. For every $x \in X$, choose a point $y \in \operatorname{bd} K$ such that $P(y) = x$. These points y form the desired set in $\mathbb{R}^d$. □

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