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NOTE

The Chromatic Number of the Graph of Large Distances

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Let S be a set of n points in $\mathbb{R}^d$. Let $d_1 > d_2 > \ldots > d_k > \ldots$ be the distances between the points in S. We assign the following graph $G(S, \leq k)$ to the set S. The vertices of $G(S, \leq k)$ correspond to the points in S. Two vertices are connected iff the distance of the corresponding points is at least d_k. The authors (1987) studied the chromatic number $\chi(G(S, \leq k))$ of this graph in the plane. It was shown that $\chi(G(S, \leq k)) = O(k^2)$ for any set S. Furthermore, if n is large enough ($n > \text{const} \cdot k^2$) then $\chi(G(S, \leq k)) \leq 7$ (independently of k). Similar results were proved in the case when the points of S form the vertices of a convex polygon: then $\chi(G(S, \leq k)) \leq 3k$, and if n is large enough then $\chi(G(S, \leq k)) \leq 3$.

In this paper we study the problem in higher dimension. We give a construction that shows that without some further assumptions on the points of S the chromatic number of $G(S, \leq k)$ can not be bounded independently of k even in 3 dimensions, no matter how large is n. Then we show that if we assume that no d of the points of S are contained in a $(d-2)$-dimensional affine subspace, and n is large enough, then the chromatic number of $G(S, \leq k)$ is bounded by a function $f(d)$ of the dimension, and show that the best value of $f(d)$ is determined by a basic number in discrete geometry.

The case when S consists of the vertices of a convex polytope seems to be more difficult and we can only offer some remarks. Even the case $k = 1$ is unsolved: The conjecture that it is at most $d+1$ is the discrete (and quite possibly most difficult) version of the famous conjecture of Borsuk (1933). (A sufficiently dense set on a sphere shows that this bound is can certainly not be improved.) Maybe the following is true and even provable without settling first the Borsuk conjecture (the results mentioned above show that this is true for $d = 2$):

Conjecture. Let $h(d)$ denote the least number for which every compact set in $\mathbb{R}^d$ with diameter 1 can be partitioned into $h(d)$ sets with diameter less than 1. Let $k \geq 1$ and let S consist of the vertices of a convex polytope in $\mathbb{R}^d$. Assume that $|S|$ is large enough (depending on d and k). Then $\chi(G(S, \leq k)) \leq h(d)$.

Let us return to the "non-convex" case. We start with a construction.

Construction 1. Let the points u_i $(1 \leq i \leq n - k/2)$ be in the xy-coordinate plane on a unit circle about the origin. Let the z_i $(1 \leq i \leq k/2)$ be points on the z-axis, where $z_i = (0, 0, 4i)$ $(1 \leq i \leq k/2)$. Then the edges of $G(S, \leq k)$ are of the following types: (z_i, z_j), which determine $k/2$ different distances, and (u_j, z_i) where for a given z_i, all the $n - k/2$ edges are of the same length, and which therefore determine $k/2$ further distances. All the other distances are at most 2, so the k largest distances remain the same for arbitrarily large n. Furthermore, the vertices z_i form a complete graph on $k/2$ vertices and the u_j's are independent, so $\chi(G(S, \leq k)) = k/2 + 1$. ∎

Analyzing the above construction, one can notice that there are $k/2$ points on a line. So one may hope that if the points of S are in general position, the chromatic number of $G(S, \leq k)$ can be bounded. The following theorem shows that this is indeed the case.

Let $g(d)$ denote the least number of parts into which the d-dimensional unit ball can be cut so that the diameter of each part is at most 1. (For estimates on this number, see Erdős and Rogers (1962).)

Theorem. *Let S be a set of n points in $\mathbb{R}^d$ such that no d of its elements are contained in a $d-2$-dimensional subspace and assume that*

$$n > 2(d+1)^d k^d.$$

Then

$$\chi(G(S, \leq k)) \leq g(d) + d - 1.$$

Moreover, for every $d \geq 2$ there exists a k and there exist arbitrarily large (even infinite) sets of points in $\mathbb{R}^d$ such that no d of their elements are contained in a $d-2$-dimensional subspace and

$$\chi(G(S, \leq k)) = g(d) + d - 1.$$

Proof. Let $S_0 = S$ and define the points p_i and the sets S_i, $(1 \leq i \leq d-1)$ in the following way: Let p_i $(1 \leq i \leq d-1)$ be the point of S connected to the largest possible number of points in S_{i-1}, and let S_i be the set of these points in S_{i-1}.

Let $S^* = S\setminus\{p_1, \ldots, p_{d-1}\}$. Let C be a ball of smallest possible radius r which contains the points of S^*. If $r < d_k$ then we construct the coloration easily as follows. Cut the ball C to $g(d)$ pieces of diameter at most r. These pieces contain independent vertices of the graph $G(S, \le k)$, so they may get one colour and the remaining p_i's can get the remaining $d-1$ colors.

So we may suppose that $r \ge d_k$. It is clear that the convex hull of the points of S^* on the surface of C contains the center c of C. Hence by Carathéodory's Theorem, there exists a set $T \subset S^*$ of at most $d+1$ elements such that already the convex hull of T contains the center of C. Now it is trivial to see that each point q in the space is at a distance r or more from some point in T. (Consider the hyperplane through c perpendicular to the vector $q - c$: at least one points $t \in T$ must be on the side of this hyperplane opposite to q, and then clearly the distance of t and q is at least r.) So every point of S is connected to at least one point of T in the graph $G(S, \le k)$.

Let us notice that

$$|S_i| \ge \frac{1}{d+1}|S_{i-1}|,$$

because p_i was chosen in such a way that it was connected to a maximum number of points in S_{i-1} and there is at least one point in T which is connected to at least $\frac{1}{d+1}|S_{i-1}|$ points. Since this holds for every i, we have

$$|S_{d-1}| \ge \frac{n}{(d+1)^{d-1}}$$

and hence there exists a $t \in T$ from which at least $n/(d+1)^d$ edges go to S_{d-1}. So S_{d-1} contains $n/(d+1)^d$ points which are connected to each of $p_1, \ldots, p_{d-1}$ and t. Now each of these points is a point of intersection of d balls, with centers $p_1, \ldots, p_{d-1}$ and t and the radii chosen from the k largest distances in S. Furthermore, the intersection of d balls whose centers are in general position can consists of at most 2 points. So we can get at most $2k^d$ intersection points, but this contradicts the assumption that t was connected to more than $n/(d+1)^d$ points in S_{d-1}, if $n > 2(d+1)^d k^d$. So our assumption that $r \ge d_k$ led to a contradiction.

To complete the proof of the theorem we construct a set that shows the tightness of our bound.

Construction 2. By the definition of $g(d)$, if we cut the unit ball into $g(d) - 1$ pieces, then at least one piece has diameter larger than 1.

One can easily show that there exists an $\epsilon > 0$ such that if we cut the unit ball into $g(d) - 1$ pieces, then at least one piece has diameter at least $1 + \epsilon$. Define the graph

$$G_n = \{(x, y) : \rho(x, y) \ge 1 + \frac{1}{n}\}$$

and let $G = \cup G_n$. We have $\chi(G) \geq g(d)$ and so by the Erdős–de Bruijn Theorem, G has a finite subgraph G' with the same chromatic number. But then G' is contained in G_n for some n, and so $\chi(G_n) \geq g(d)$. So $\epsilon = 1/n$ can be chosen.

So if we form the graph whose points are the points of the unit ball, two being connected if and only if their distance is at least $1+\epsilon$, then the chromatic number of this graph is at least $g(d)$. By the theorem of Erdős and de Bruijn (1951), this graph has a finite subgraph with chromatic number at least $g(d)$. Let S_0 be the set of vertices of this finite subgraph, and let q_0 denote the center of the unit ball in consideration.

Next, construct a regular $(d-1)$-dimensional simplex with vertices $q_0, q_1, \ldots, q_{d-1}$ with all sides 3. Let k denote the number of distinct distances not smaller than $1+\epsilon$ determined by the set $S_0 \cup \{q_1, \ldots, q_{d-1}\}$. Let A be the set of points which are at a distance of 3 from $q_1, \ldots, q_{d-1}$ and at a distance less than ϵ from q_0 (A is a little arc of a circle). Let A' be any subset of A and $S = S_0 \cup \{q_1, \ldots, q_{d-1}\} \cup A'$. Note that all the new distances created by adding the points in A' are shorter than $1+\epsilon$, and hence the k largest distances among the points of S are precisely those distances not smaller than $1+\epsilon$.

We claim that $\chi(G(S, \leq k)) \geq g(d) + d - 1$. For, consider any coloration of this graph. The points $q_1, \ldots, q_{d-1}$ must get different colors and these colors must also differ from the colors of the rest of the points. Moreover, the points in S must get at least $g(d)$ distinct colors by the choice of S. This completes the proof. ■

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