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The Book-Tree Ramsey Numbers

Paul Erdös, R.J. Faudree, C.C. Rousseau, and R.H. Schelp*

Dedicated to Professor Roberto W. Frucht on his 80th birthday

Abstract. In 1978 Rousseau and Sheehan showed that the book-star Ramsey number

$$r(K(1,1,m), K_{1,n-1}) = 2n-1 \quad \text{for} \quad n \geq 3m-3.$$

We show that this result is true when the star is replaced by an arbitrary tree on n vertices.

I. Preliminaries.

Let G_1 and G_2 be simple graphs without isolated vertices. The *Ramsey number* $r(G_1, G_2)$ is the smallest positive integer p such that coloring each edge of K_p one of two colors there is either a copy of G_1 in the first color or a copy of G_2 in the second color. By tradition, we shall let the colors be $\mathcal{R}$ (red) and $\mathcal{B}$ (blue) with the resulting edge-induced subgraphs denoted $\langle\mathcal{R}\rangle$ and $\langle\mathcal{B}\rangle$ respectively. Throughout the paper a colored K_p will always refer to one in which each edge is colored red or blue.

It is well known for a connected graph G_2 that

$$(1) \qquad r(G_1, G_2) \geq (\chi(G_1) - 1)(p(G_2) - 1) + s(G_1), \quad p(G_2) \geq s(G_1),$$

where $\chi(G_1)$ is the chromatic number of G_1, $p(G_2)$ the order of G_2, and $s(G_1)$ the chromatic surplus of G_1. Here the *chromatic surplus* is the smallest number of vertices in a color class under any $\chi(G_1)$-coloring of the vertices of G_1. Inequality (1) follows by coloring red or blue the edges of a complete graph on $(\chi(G_1)-1)(p(G_2)-1)+s(G_1)-1$ vertices such that the blue graph $\langle\mathcal{B}\rangle$ is isomorphic to $(\chi(G_1)-1)K_{p(G_2)-1} \cup K_{s(G_1)-1}$ and the red graph $\langle\mathcal{R}\rangle$ is isomorphic to its complement. Of interest is the case when inequality (1) is in fact an equality.

Let T_n denote a tree on n vertices and let B_m denote the graph $K(1,1,m)$ called an *m-book* or a *book with m pages*. In this paper we investigate when equality holds in (1) with $G_1 = B_m$ and $G_2 = T_n$, i.e., when $r(B_m, T_n) = 2n-1$. The more general problem when G_1 is the multipartite graph $K(1,1,m_1,m_2,...,m_k)$ and $G_2 = T_n$ with *n large* has been considered in [**2**]. In fact the value of $r(K(m_1,m_2,...,m_k), T_n)$ with *n large* has received considerable attention (see [**3,4**]).

The following notation will be used. If the graph G has at least (at most) ℓ vertices of a given type, or order at least (at most) ℓ, we write $\geq \ell (\leq \ell)$. This symbolism is adopted to avoid frequent usage of the words 'at least' or 'at most'. As is customary $\lceil x \rceil$ ($\lfloor x \rfloor$) will denote the least (greatest) integer $\geq x\,(\leq x)$. Additional notation will follow that used in standard texts, e.g., [**1,5**].

Since in this paper we wish to show $r(B_m, T_n) = 2n-1$ (for a certain range of values of m and n) and from (1) $r(B_m, T_n) \geq 2n-1$, it will be assumed throughout that equality follows from showing $r(B_m, T_n) \leq 2n-1$.

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II. The Book Star Ramsey Number.

In [6] it is shown that $r(B_m, K_{1,n-1}) = 2n-1$ when $n \geq 3m-3$. Our main objective is to show that the star $K_{1,n-1}$ can be replaced by an arbitrary tree T_n with the same result. The lengthy argument needed to prove this fact will be deferred to the next section. In this section we wish to first establish that there is no hope to prove (in general) that $r(B_m, K_{1,n-1}) = 2n-1$ for $n < 3m-3$. To see this we introduce a *rectangular coloring* of the edges of K_p into the classes $\mathcal{R}$ and $\mathcal{B}$ as follows: partition $V(K_p) = \{X_{11}, ..., X_{MN}\}$ and set $[X_{ij}]^2 \subseteq \mathcal{B}$, and

$$[X_{ij}, X_{i'j'}] \subseteq \begin{cases} \mathcal{B} & \text{if } i = i' \text{ or } j = j' \\ \mathcal{R} & \text{otherwise.} \end{cases}$$

Set $M = 3$, $N = a$, and $|X_{ij}| = b$ for all i and j. It is easy to check that a rectangular coloring of $E(K_{3ab})$ in which both

$$(a+2)b \leq n-1 \quad \text{and} \quad (a-2)b \leq m-1 \tag{2}$$

contains no red B_m and no blue $K_{1,n-1}$. If a and b are chosen such that the inequalities of (2) hold and $3ab \geq 2n-1$, then $2ab \leq n+m-2$ and $2n-1 \leq 3ab \leq \frac{3}{2}(n+m-2)$ so that $n \leq 3m-4$. In such cases the rectangular coloring shows

$$r(B_m, K_{1,n-1}) \geq 3ab+1 > 2n-1 \quad \text{with} \quad n \leq 3m-4.$$

For the readers sake we include the counting argument of Rousseau and Sheehan which proves the book-star Ramsey number mentioned earlier.

Theorem 1[6]. *The Ramsey number $r(B_m, K_{1,n-1}) = 2n-1$ for $n \geq 3m-3$.*

Proof: Color K_{2n-1} such that $\langle \mathcal{B} \rangle$ contains no $K_{1,n-1}$. Then the red degree of x, $d_{\mathcal{R}}(x)$, satisfies $d_{\mathcal{R}}(x) \geq (2n-2)-(n-2) = n$ for all vertices x. Thus $\langle \mathcal{R} \rangle$ contains a K_3. Let $\{a,b,c\}$ be the set of vertices of this red K_3 and let $N_{\mathcal{R}}(a), N_{\mathcal{R}}(b)$, and $N_{\mathcal{R}}(c)$ denote the red neighbors of a, b and c respectively. Further set $A = N_{\mathcal{R}}(a) - \{b,c\}$, $B = N_{\mathcal{R}}(b) - (N_{\mathcal{R}}(a) \cup \{a\})$ and $C = N_{\mathcal{R}}(c) - (N_{\mathcal{R}}(a) \cup N_{\mathcal{R}}(b))$. If $\langle \mathcal{R} \rangle$ contains no B_m, then each of the following inequalities hold: $|A| \geq n-2$, $|B| \geq (n-2)-(m-2) = n-m$ and $|C| \geq (n-2)-2(m-2) = n-2m+2$. Thus if both $\mathcal{R} \not\supseteq B_m$ and $\langle \mathcal{B} \rangle \not\supseteq K_{1,n-1}$, then

$$\begin{aligned} 2n-1 = |V(K_{2n-1})| &\geq |\{a,b,c\} \cup A \cup B \cup C| \\ &\geq 3+(n-2)+(n-m)+(n-2m+2). \end{aligned}$$

This gives $n \leq 3m-4$, a contradiction, and completes the proof.

III. The Book-Tree Ramsey Number.

As mentioned earlier the main objective of the paper is to prove that Theorem 1 holds when the star $K_{1,n-1}$ is replaced by an arbitrary tree T_n. The proof of this is lengthy and will be accomplished first for a special case and then in general through the use of a collection of lemmas.

Theorem 2. *The Ramsey number $r(B_m, T_n) = 2n - 1$ for $n \geq 3m - 3$.*

Before proving this theorem when the tree T_n satisfies a special condition we give a useful lemma.

Lemma 3. *Let K_t be colored such that $\langle \mathcal{R} \rangle \not\supseteq B_m$ and $\langle \mathcal{B} \rangle \not\supseteq T_n$. Then the red degree of each of its vertices is $\leq n + m - 2$.*

Proof: Suppose there exists a vertex of red degree $\geq n + m - 1$. Let this vertex be v and its red neighborhood $N_{\mathcal{R}}$. If each vertex in $N_{\mathcal{R}}$ has $\geq n - 1$ blue adjacencies in $N_{\mathcal{R}}$, then the tree T_n can be constructed in $N_{\mathcal{R}}$ using only vertices of $N_{\mathcal{R}}$. Hence there exists a vertex w in $N_{\mathcal{R}}$ that has $\geq (n + m - 2) - (n - 2) = m$ red adjacencies in $N_{\mathcal{R}}$. But then v and w are red adjacent and are commonly red adjacent to m vertices, contradicting $\langle \mathcal{R} \rangle \not\supseteq B_m$.

Proposition 4. *Theorem 2 holds when $\Delta(T_n) \geq \frac{2}{3}n$.*

Proof: Let K_{2n-1} be colored and suppose $\langle \mathcal{R} \rangle \not\supseteq B_m$ and $\langle \mathcal{B} \rangle \not\supseteq T_n$. By the last lemma each vertex in the colored graph has blue degree $\geq (2n - 2) - (n + m - 2) = n - m \geq n - (n + 3)/3 = \frac{2}{3}n - 1$. Also by Theorem 1 $\langle \mathcal{B} \rangle$ contains a star on $n - 1$ edges. Let x denote the center of this star. Further let y be the vertex of largest degree in T_n, and let A denote the set of endvertices of T_n adjacent to y.

We first show that the subtree $T' = \langle V(T_n) - A \rangle$ of T_n can be embedded in $\langle \mathcal{B} \rangle$. Start this embedding by mapping y to x and extend this map to a maximal subtree T'' of T' in $\langle \mathcal{B} \rangle$. Observe, since the blue degree of each vertex of the colored graph is $\geq \frac{2}{3}n - 1$, that T'' contains $\geq \frac{2}{3}n$ vertices. Also since y is not adjacent to any endvertices of T'', $\geq (\frac{2}{3}n - 1)/2 \geq (n - 2)/3$ of these vertices of T'' are non-neighbors of y. But the degree of y is $\geq \frac{2}{3}n$ so y has $\leq (n - 1) - (\frac{2}{3}n) = \frac{1}{3}n - 1$ non-neighbors in T'. Hence $T'' = T'$ and T' is embedded in $\langle \mathcal{B} \rangle$ with y mapped to x.

The embedding is easily extendable in $\langle \mathcal{B} \rangle$ to all of T_n, since x has $n-1$ blue neighbors. This contradicts the supposition $\langle \mathcal{B} \rangle \not\supseteq T_n$, completing the proof.

For the remainder of this section we will assume that T_n fails to satisfy the condition of Proposition 4. Before we continue towards a complete proof of Theorem 2 we outline the strategy followed. Assuming the colored graph K_{2n-1} contains neither a red B_m nor a blue T_n, we will show $V(K_{2n-1})$ contains disjoint sets X and Y such that $\langle X \rangle$ contains all blue forests of order $\leq \lceil \frac{2}{3}n \rceil$ and $\langle Y \rangle$ contains all blue forests of order $\leq \lceil \frac{1}{3}n \rceil$. Furthermore these forests can be embedded such that each component can be rooted arbitrarily. Next we show that the tree can be split appropiately to fit its 'parts' into the blue graphs of $\langle X \rangle$ and $\langle Y \rangle$, and these parts can be connected by blue edges from X to Y. This is the essential content of the next three lemmas needed in the proof of Theorem 2.

Lemma 5. *Let K_{2n-1} be colored such that $\langle \mathcal{R} \rangle \not\supseteq B_n$ and $\langle \mathcal{B} \rangle \not\supseteq T_n$. Then there exist disjoint sets of vertices X and Y in the colored graph, $|X| \geq n$, $|Y| \geq n - m + 1$, such that the blue degree of each vertex in $\langle X \rangle$ is $\geq n - m$ and the blue degree of each vertex in $\langle Y \rangle$ is $\geq n - 2m + 1$.*

Proof: Among all vertices choose one, say w, of largest red degree. Let X denote that set of red neighbors of w. To see that $|X| \geq n$ build a largest subtree T of T_n in $\langle \mathcal{B} \rangle$. Since

T is a proper subgraph of T_n there exists a vertex n of T with all its blue adjacencies to other vertices of T. Thus v has $(2n-1)-(n-1)=n$ red adjacencies and $|X| \geq n$. For convenience assume $|X| = n+t$ with $t \geq 0$.

Since $\langle \mathcal{R} \rangle \not\supseteq B_m$ each vertex of X has $\geq (n+t-1)-(m-1) = n+t-m$ blue adjacencies in X. Using this blue degree build a largest blue subtree T' of T_n in $\langle X \rangle$ and extend T to a largest blue subtree T' of T_n avoiding vertex w. Note that T' contains all but $\leq m-1$ vertices of X.

Since T' is a proper subgraph of T_n, one of its endvertices, say z, is red adjacent to $\geq (2n-1)-(n-1)=n$ vertices with $m-1$ of them in X. Hence z is red adjacent to $\geq n-m+1$ vertices not in X. Let Y denote this set of $\geq n-m+1$ red neighbors of z lying outside of X. Since $\langle \mathcal{R} \rangle \not\supseteq B_m$ each vertex in Y has $m-1$ red adjacencies in Y, so that each such vertex has $\geq n-2m+1$ blue degree in $\langle Y \rangle$.

Lemma 6. *One of the following occurs.*

(i) There exists an edge e of T_n such that the two components of $T_n - e$ have orders $\lceil \frac{2}{3}n \rceil$ and $\lfloor \frac{1}{3}n \rfloor$ respectively.

(ii) There exists a vertex v of T_n such that the components of $T_n - v$ of order $\leq \lfloor \frac{1}{3}n \rfloor$ contains $\geq \lfloor \frac{1}{3}n \rfloor$ vertices of T_n.

Proof: Assume (i) does not occur. For $e = vw$ an edge of T_n let C_v and C_w denote the components of $T_n - e$ containing vertex v and w respectively. Choose e such that C_v is of minimal order with $|V(C_v)| > \lceil \frac{2}{3}n \rceil$ and $|V(C_w)| < \lfloor \frac{1}{3}n \rfloor$. From the minimality of the order of C_v it is clear that $d(v) > 2$. Thus let $C_1, C_2, ..., C_s$ be the components of $C_v - v$ with each v_i in C_i and adjacent to v and $|V(C_1)| \geq |V(C_2)| \geq ... \geq |V(C_s)|$. If $\lfloor \frac{1}{3}n \rfloor \geq |V(C_1)|$, then (ii) follows. Thus assume $|V(C_1)| > \lfloor \frac{1}{3}n \rfloor$. Since (i) does not occur we can assume $d(v_1) > 2$. Let $C_1', C_2', ..., C_\ell'$ be the components of $C_1 - v_1$ in C_1 with each w_i in C_i' and adjacent to v_1 and $|V(C_1')| \geq |V(C_2')| \geq ... \geq |V(C_\ell')|$. If $\lfloor \frac{1}{3}n \rfloor \geq |V(C_1')|$, then (ii) follows by replacing v by v_1, while if $|V(C_1)| > \lfloor \frac{1}{3}n \rfloor$ repeat the last argument replacing C_1 by C_1' and v_1 by w_1. After an appropriate number of repetitions (ii) occurs.

Lemma 7. *One of the following occurs.*

(i) There exists an edge e of the tree T_n such that the order of each of the components of $T_n - e$ is $\leq \lceil \frac{2}{3}n \rceil$.

(ii) There exists a vertex v of the tree T_n such that the order of each of the components of $T_n - v$ is $\leq \lfloor \frac{1}{3}n \rfloor$.

Proof: Assume (i) does not occur. As in the proof of the last lemma, for $e = uv$ an edge of T_n, let C_v and C_w be the components of $T_n - e$ containing v and w respectively. Choose e such that C_v is of minimal order with $|V(C_n)| > \lceil \frac{2}{3}n \rceil$ and $|V(C_w)| < \lfloor \frac{1}{3}n \rfloor$. Thus $d(v) > 2$. Let $v_1, v_2, ..., v_s$ be the vertices (other than w) adjacent to v. Denote by $C_1, C_2, ..., C_s$ the components of $C_v - v$ with $v_i \in V(C_i)$ for each i. From the minimality of $|V(C_n)|$ it follows that $|V(C_i)| \leq \lceil \frac{2}{3}n \rceil$ for all i. Also if $|V(C_j)| > \lfloor \frac{1}{3}n \rfloor$ for some j, then the components of $T - v_j v$ would satisfy (i). Hence the components of $T_n - v$ satisfy the condition given in (ii).

We are now in a position to complete the proof of Theorem 2.

Proof of Theorem 2: Again suppose that the graph K_{2n-1} has been colored such that $\langle \mathcal{R} \rangle \not\supseteq B_m$ and $\langle \mathcal{B} \rangle \not\supseteq T_n$. By Lemma 5 there exists disjoint sets X and Y in the colored graph, $|X| \geq n$, $|Y| \geq n - m + 1 \geq n - (\frac{1}{3}n + 1) + 1 = \frac{2}{3}n$, such that the blue degree of each vertex in $\langle X \rangle$ is $\geq n - m \geq \frac{2}{3}n - 1$ and the blue degree of each vertex in $\langle Y \rangle$ is $\geq n - 2m + 1 \geq \frac{1}{3}n - 1$. For $x \in X$ and $y \in Y$ we denote these blue degrees by $d_{X,\mathcal{B}}(x)$ and $d_{Y,\mathcal{B}}(y)$ respectively. More generally for each vertex z and each set of vertices W we let $d_{W,\mathcal{B}}(z)$ denote the number of blue adjacencies of z in W.

Since $|X| \geq n$ and $\langle \mathcal{B} \rangle \not\supseteq T_n$, there exists a pair of vertices $x_1, x_2 \in X$ that are red adjacent. But $\langle \mathcal{R} \rangle \not\supseteq B_m$ so that either $d_{Y,\mathcal{B}}(x_1)$ or $d_{Y,\mathcal{B}}(x_2)$ is $\geq (|Y| - (m-1))/2 = (n - 2m + 2)/2 \geq n/6$. Without loss of generality assume $d_{Y,\mathcal{B}}(x_1) \geq n/6$. Also from the blue degrees of vertices in $\langle X \rangle$ and $\langle Y \rangle$ calculated above, it is clear that $\langle X \rangle$ ($\langle Y \rangle$) contains an arbitrary forest in $\mathcal{B}$ of order $\leq \lceil \frac{2}{3}n \rceil$ ($\leq \lceil \frac{1}{3}n \rceil$) with all components rooted arbitrarily.

From Proposition 4 we assume throughout the proof that $\Delta(T_n) < \frac{2}{3}n$. We break the remainder of the proof into two cases.

Case 1: There exists a vertex v in T_n such that the largest $\lceil n/6 \rceil$ components of $T_n - v$ of order $\leq \frac{1}{3}n$ contain collectively $\geq \lfloor \frac{1}{3}n \rfloor$ vertices.

Let $C_1, C_2, ..., C_\ell$ be the components of $T_n - v$ of order $\leq \frac{1}{3}n$ with $|V(C_1)| \geq |V(C_2)| \geq ... \geq |V(C_\ell)|$. We show T_n can be embedded in the blue subgraph of $\langle X \cup Y \rangle$.

Embed v at x_1 and since $d_{Y,\mathcal{B}}(x_1) \geq n/6$, continue to embed sequentially all vertices of components $C_1, C_2, ..., C_{\lceil n/6 \rceil}$ in the blue subgraph of $\langle Y \rangle$ until all these vertices are embedded or until the embedding stops. In the embedding procedure we only use blue neighbors of x_1 in Y if no other choices are available. If all the vertices of these components are embeddable in $\langle Y \rangle$, being $\geq \lfloor \frac{1}{3}n \rfloor$ in number, the remaining vertices of the tree are embeddable in the blue subgraph of $\langle X \rangle$. Thus assume in this embedding all vertices of $C_1, C_2, ..., C_j$ have been embedded ($j \geq 1$) and that the embedding stops at some vertex w_1 of C_{j+1}. If $|\bigcup_{i=1}^{j} V(C_i)| \geq \lfloor \frac{1}{3}n \rfloor$, then the remainder of the tree $T_n - \bigcup_{i=1}^{j} C_i$ can be embedded in the blue subgraph of $\langle X \rangle$.

Thus we assume $|\bigcup_{i=1}^{j} V(C_i)| < \lfloor \frac{1}{3}n \rfloor$ and that the embedding of the next component C_{j+1} in the blue subgraph of $\langle Y \rangle$ stops at some vertex w_1. Continue this embedding to a largest subtree T of C_{j+1} in the blue subgraph of $\langle Y \rangle$. This gives a collection of endvertices $w_1, w_2, ..., w_s$ of T which are red adjacent to all vertices of $Y - (V(C_1) \cup V(C_2) \cup ... \cup V(C_j) \cup V(T))$. Extend this embedding to vertices of X in $\langle \mathcal{B} \rangle$. Recall $d_{X,\mathcal{B}}(x) \geq \frac{2}{3}n - 1$ for $x \in X$ and $|V(C_{j+1})| < \lfloor \frac{1}{3}n \rfloor$, so that the remainder of C_{j+1}, namely $C_{j+1} - T$, is embeddable in the blue subgraph of $\langle X \rangle$ or this embedding stops at some $w_u, 1 \leq u \leq s$. But $d_{Y,\mathcal{B}}(y) \geq \frac{1}{3}n - 1$ implies $|V(C_1) \cup V(C_2) \cup ... \cup V(C_j) \cup V(T)| \geq \lfloor \frac{1}{3}n \rfloor$. Hence the embedding can be extended to all T_n, if the remainder of C_{j+1} is embeddable in the blue subgraph of $\langle X \rangle$. Thus the embedding stops at vertex w_u and w_u is red adjacent to all vertices of

$$(X - \{x_1\}) \cup [Y - (V(C_1) \cup V(C_2) \cup ... \cup V(C_j) \cup V(T))].$$

Letting $a = |V(C_1) \cup V(C_2) \cup ... \cup V(C_j) \cup V(T)|$ this implies w_u has $\geq (|X| - 1) + |Y| - a$ red adjacencies.

From the proof of Lemma 5 we can assume that no vertex in the colored K_{2n-1} graph has red degree $> |X|$. Thus $a \geq |Y| - 1 \geq \frac{2}{3}n - 1$. But by assumption $|\bigcup_{i=1}^{j} V(C_i)| < \lfloor \frac{1}{3}n \rfloor$ so $|V(C_{j+1})| > |V(T)| \geq (\frac{2}{3}n - 1) - (\frac{1}{3}n - 1) = \frac{1}{3}n$ a contradiction to $|V(C_{j+1})| < \lfloor \frac{1}{3}n \rfloor$. This contradiction completes the proof in this case.

Case 2: Case 1 does not occur.

We first establish for each vertex v in T_n that the largest component of $T_n - v$ is of order $> \lfloor \frac{1}{3}n \rfloor$. Let there be t *nontrivial* components in $T_n - v$. Then if each component is of order $\leq \lfloor \frac{1}{3}n \rfloor$, it follows from the fact Case 1 does not occur that $t < \lceil n/6 \rceil$ and that these nontrivial components collectively contain $\leq \lfloor \frac{1}{3}n \rfloor - 1$ elements. Hence $\Delta(T_n) \geq (n-1) - (\lfloor \frac{1}{3}n \rfloor - 1) \geq \frac{2}{3}n$, a contradiction. This establishes what we need, namely, for each vertex v in T_n the largest component of $T_n - v$ is of order $> \lfloor \frac{1}{3}n \rfloor$.

Next observe that if there is an edge $e = zw$ in T_n such that the components of $T_n - e$ have orders $\lceil \frac{2}{3}n \rceil$ and $\lfloor \frac{1}{3}n \rfloor$, respectively, then T_n is embeddable in $\langle \mathcal{B} \rangle$. This follows by mapping e to any blue edge from x_1 to the set Y, and embedding the large component of $T_n - e$ in the blue subgraph of $\langle X \rangle$ rooted at x_1 and the smaller component of $T_n - e$ in the blue subgraph of $\langle Y \rangle$ appropiately rooted. We therefore assume that Lemma 6 (ii) holds.

Let v the vertex of T_n guaranteed by Lemma 6 (ii) and let $T_n - v$ have components $C_1, C_2, ..., C_\ell$ with $|V(C_1)| \geq |V(C_2)| \geq ... \geq |V(C_\ell)|$. Since Lemma 6 (ii) holds, $|V(C_1)| \leq \lceil \frac{2}{3}n \rceil$ and by what was earlier established $\lfloor \frac{1}{3}n \rfloor < |V(C_1)|$. Also we may assume $\langle Y \rangle$ contains a red edge, otherwise the blue graphs of both $\langle X \rangle$ and $\langle Y \rangle$ contain any rooted blue tree of order $\leq \lceil \frac{2}{3}n \rceil$ which by Lemma 7 implies T_n is embeddable in the blue graph of $\langle X \cup Y \rangle$. Let $y_1 y_2$ be a red edge of $\langle Y \rangle$.

Since $\langle \mathcal{R} \rangle \not\supseteq B_m$, either $d_{\langle \overline{X} \rangle, \mathcal{B}}(y_1)$ or $d_{X,\mathcal{B}}(y_2)$ is $\geq (|X| - (m-1))/2 = (n-m+1)/2 \geq \frac{1}{3}n$. Assume $d_{X,\mathcal{B}}(y_1) \geq \frac{1}{3}n$.

Consider the vertex v of T_n and the components $C_1, C_2, ..., C_\ell$ of $T_n - v$ given above with $\lceil \frac{2}{3}n \rceil \geq |V(C_1)| \geq |V(C_2)| \geq ... \geq |V(C_\ell)|$ and $|V(C_1)| > \lfloor \frac{1}{3}n \rfloor$. Map v to y_1, embedding C_1 in the blue subgraph of $\langle X \rangle$ such that a minimal number of blue adjacencies of y_1 to elements of X are used. Since Case 1 fails to hold, the total number of vertices in the set of nontrivial components of $T_n - v$ of order $\leq \frac{1}{3}n$ is $< \lfloor \frac{1}{3}n \rfloor$. But Lemma 6 (ii) holds so by including an appropriate number of trivial components of $T_n - v$ with all those nontrivial ones of order $\leq \frac{1}{3}n$, we find a set of vertices with exactly $\lfloor \frac{1}{3}n \rfloor$ elements which can be embedded in the blue subgraph of $\langle Y \rangle$ and which extends the embedding of $< \{v\} \cup V(C_1) >$ described above. Since $|V(C_1)| > \lfloor \frac{1}{3}n \rfloor$, $d_{X,\mathcal{B}}(y_1) \geq \frac{1}{3}n$ and the blue subgraph of $\langle X \rangle$ contains all forests of order $\leq \lceil \frac{2}{3}n \rceil$ with arbitrarily rooted components, the given embedding can be extended in the blue subgraph of $\langle X \cup Y \rangle$ to include all of T_n, a contradiction.

This final contradiction completes the proof of Case 2 and the proof of Theorem 2.

From Theorem 2 a more general result can be proved by induction.

Theorem 8. *The Ramsey number* $r(K_\ell + \overline{K}_m, T_n) = \ell(n-1) + 1$ *for* $\ell \geq 2$ *and* $n \geq 3m - 3$.

Proof: The usual canonical example shows $r(K_\ell + \overline{K}_m, T_n) \geq \ell(n-1) + 1$. Thus color each edge of a $K_{\ell(n-1)+1}$ red or blue. By Theorem 2 the result follows for $\ell = 2$. Thus assume $\ell > 2$ and that the result holds for all values $< \ell$.

Build the largest order subtree T of T_n in $\langle \mathcal{B} \rangle$. If T is a proper subgraph of T_n, then there exists a vertex v of T of red degree $\geq (\ell - 1)(n-1) + 1$. Denote this set of red adjacencies of v by $N_{\mathcal{R}}$. But since $\langle \mathcal{B} \rangle \not\supseteq T_n$, the red subgraph of $\langle N_{\mathcal{R}} \rangle$ contains by assumption the graph $K_{\ell-1} + \overline{K}_m$. This red $K_{\ell-1} + \overline{K}_m$ with vertex v span a red $K_\ell + \overline{K}_m$, completing the inductive proof.

IV. Conclusion

The rectangular coloring given in Section II showed that $r(B_m, T_n) > 2n - 1$ for certain $n \leq 3m - 4$. It is in fact shown in [6] that $r(K_\ell + \overline{K}_m, T_n) \leq \ell(n-1) + m$ and that equality holds when $n - 1$ divides $m - 1$. Thus it is of particular interest to learn more about $r(B_m, T_n)$ whenever $m < n \leq 3m - 4$.

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P. Erdős, Hungarian Academy of Sciences, P.O.Box 127, Budapest, Hungary H-1364
R.J. Faudree, C.C. Rousseau, and R.H. Schelp, Department of Mathematical Sciences, Memphis State University, Memphis, Tennessee 38152, U.S.A.