K-PATH IRREGULAR GRAPHS

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ABSTRACT

A connected graph G is k-path irregular, $k \geq 1$, if every two vertices of G that are connected by a path of length k have distinct degrees. This extends the concepts of highly irregular (or 2-path irregular) graphs and totally segregated (or 1-path irregular) graphs. Various sets S of positive integers are considered for which there exist k-path irregular graphs for every $k \in S$. It is shown for every graph G and every odd positive integer k that G can be embedded as an induced subgraph in a k-path irregular graph. Some open problems are also stated.

1. INTRODUCTION

In [1] a connected graph was defined to be highly irregular if each of its vertices is adjacent only to vertices with distinct degrees. Equivalently, a graph G is highly irregular if every two vertices of G connected by a path of length 2 have distinct degrees. In [4] Jackson and Entringer extended this concept by considering those graphs in which every two adjacent vertices have distinct degrees. They referred to these graphs as totally segregated. Jackson and Entringer [3] noted that these are the cases $k = 2$ and $k = 1$, respectively, of the property that the end-vertices of every path of length k have different degrees.

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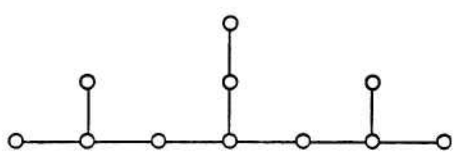
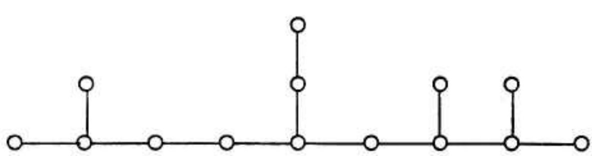
More generally, then, we define a connected graph G to be k-path irregular $k \geq 1$, if every two vertices of G that are connected by a path of length k have distinct degrees. Thus, the highly irregular graphs are precisely the 2-path irregular graphs, while the totally segregated graphs are the 1-path irregular graphs. In this paper we present some results concerning k- path irregular graphs and state some open problems.

For each positive integer k, there exists a k-path irregular graph. Of course, every graph of order at most k is k-path irregular. Indeed, any graph containing no path of length k is vacuously k-path irregular. Less trivially, the path of length $k + 1$ is k-path irregular. Even this graph contains only two paths of length k, however. We now consider k-path irregular graphs with many paths of length k.

A connected graph G is homogeneously k- path irregular if G is k-path irregular and every vertex of G is an end-vertex of a path of length k. Figure 1 shows homogeneously k-path irregular trees for $k = 3$ and $k = 4$.



A homogeneously 3-path irregular tree of order 11.



A homogeneously 4-path irregular tree of order 14.

The trees shown in Figure 1 belong to a more general class of trees. In fact, for each integer $k \geq 1$, there exists a homogeneously k-path irregular tree. The paths P_3 and P_4 are homogeneously 1-path and 2-path irregular, respectively.

while Figure 2 describes the construction of a homogeneously k-path irregular tree of order $3k + \lfloor (k+1)/2 \rfloor$ for $k \geq 3$.

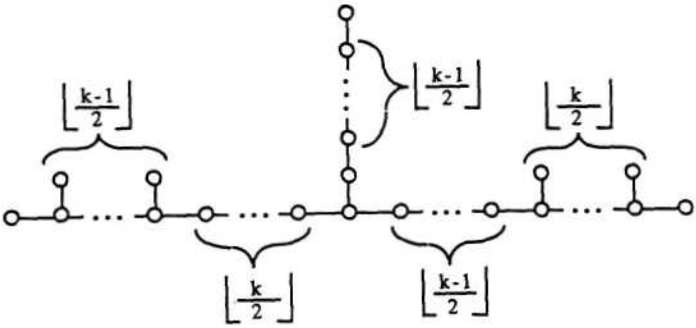


Figure 2. A homogeneously k path irregular tree for $k \geq 3$

2. Graphs That Are k-Path Irregular for Many Values of k.

We have already noted the existence of k-path irregular graphs for a given, fixed positive integer k. We now consider the existence of graphs that are k-path irregular for several values of k. First, we show that the values of k have some limitations in general.

Proposition 1. Only the trivial graph is k-path irregular for every positive integer k.

Proof: If G is a nontrivial (connected) graph, then it is well-known that G contains distinct vertices u and v having the same degree. If $d(u, v) = d$, then G is not d-path irregular.

We now investigate some proper subsets S of positive integers such that there exist graphs that are k-path irregular for every $k \in S$. In order to consider one natural choice of such a set S, we describe a class of graphs. For a positive integer n, define the graph H_n to be that bipartite graph with partite sets

$V = \{v_1, v_2, \cdots, v_n\}$ and $V' = \{v'_1, v'_2, \cdots, v'_n\}$ such that $v_k v'_j \in E(H_n)$ if and only if $i + j \geq n + 1$ (see Figure 3).

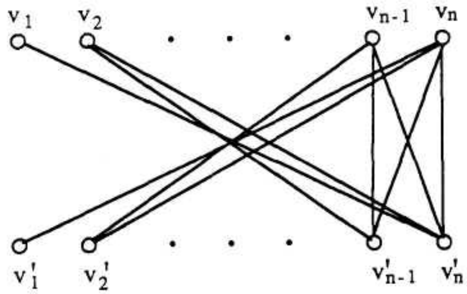


Figure 3. The graph H_n

Proposition 2. Let G be a graph with maximum degree n. If G is k-path irregular for every positive even integer k, then $G \cong H_n$.

PROOF: Let G be a graph that satisfies the hypothesis of the proposition, and suppose that $v_n \in V(G)$ such that $deg\, v_n = n$. Since G is 2-path irregular, v_n is adjacent to vertices $u_i\ (1 \leq i \leq n)$, where $deg\, u_i = i$. Similarly, u_n is adjacent to vertices $v_j\ (1 \leq j \leq n)$ with $deg\, v_j = j$. Moreover, the vertices $u_i\ (1 \leq i \leq n)$ and $v_j\ (1 \leq j \leq n)$ are distinct. For $1 \leq i < j \leq n$, the vertices u_i and u_j are not adjacent; otherwise, G contains a $u_i - v_i$ path of length 4. Similarly, no two vertices of $\{v_1, v_2, \cdots, v_n\}$ are adjacent. Further $V(G) = \{u_1, u_2, \cdots, u_n\} \cup \{v_1, v_2, \cdots, v_n\}$ since G is 4-path irregular. Thus $G \cong H_n$.

The proof given of Proposition 2 only uses a portion of the hypothesis, namely that G is k-path irregular for $k = 2$ and $k = 4$. This suggests the following problem.

Problem 1. Determine those graphs that are k-path irregular for all even integers $k \geq 4$.

A bipartite graph G with partite sets U and V is k-path irregular for all positive odd integers k, provided that $deg\ u \neq deg\ v$ for $u \in U$ and $v \in V$. Thus, $K_{m,n}$ $(m \neq n)$ is k-path irregular for all positive odd integers k. On the other hand, a graph need not be bipartite to be k-path irregular for all positive odd integers k. For example, the graph of Figure 4 has this property but is not bipartite.

Although the graph of Figure 4 is not bipartite, it does contain a bipartite block. As we shall see, this is a necessary condition for a nontrivial graph to be k-path irregular for all positive odd integers k.

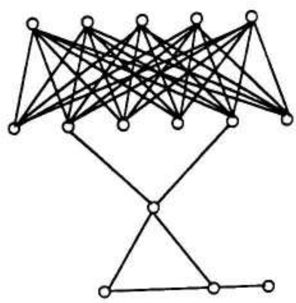


Figure 4. A non-bipartite graph that is k-path irregular for all positive odd integers k.

Proposition 3. Every nontrivial graph that is k-path irregular for all odd positive integers k contains a bipartite block.

PROOF: Let G be a graph that is k-path irregular for all odd positive integers k and assume, to the contrary, that no block of G is bipartite. Then every block contains an odd cycle and, of course, has order at least 3. It then follows that every two vertices of each block of G are connected by both a path of even length and a path of odd length. Since G is k-path irregular for every odd positive integer k, every two vertices of each block of G have distinct degrees in G.

Let B be an end-block of G (a block containing only one cut-vertex of G), and let v be the cut- vertex of G belonging to B. As we observed above, the vertices of B have distinct degrees in G. Since $deg_B u = deg_G u$ for every vertex u of B different from v and since no nontrivial graph has all of its vertices with distinct degrees, it follows that only two vertices of B have the same degree in G, and v is one of these vertices. However, then, there is a vertex $u \, (\neq v)$ in B having degree 1 (see [2]), contradicting the fact that B is a block.

We now consider the existence of graphs that are k-path irregular for $k \in S$, where S consists of a pair of consecutive positive integers.

Proposition 4. There exists a graph containing paths of length $k + 1$ that is both k- path irregular and $(k + 1)$-path irregular if and only if $k \geq 3$.

PROOF: If G is a 2-path irregular graph containing paths of length 2 with maximum degree $n \geq 2$, then G contains adjacent vertices u and v of degree n, where each of u and v is adjacent to a vertex of degree i, for each $i = 1, 2, \cdots n$. Further, these $2n$ vertices are distinct. Since u and v are adjacent vertices of degree n, the graph G cannot be 1-path irregular. Since the two vertices of degree 1 that are adjacent to u and v, respectively, are connected by a path of length 3, the graph G is not 3-path irregular. Thus, there is no graph that is both k-path irregular and $(k+1)$-path irregular for $k = 1$ or $k = 2$. Such a graph does exist, however, for $k \geq 3$, as is illustrated in Figure 5.

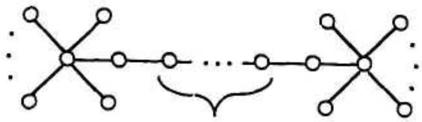


Figure 5. A graph that is both k-path irregular and $(k+1)$-path irregular for $k \geq 3$.

A natural problem is stated below.

Problem 2. Determine those positive integers ℓ and m, with $\ell < m$, for which there exists a graph that is k-path irregular for every $k \in \{\ell, \ell + 1, \cdots, m\}$.

3. Embedding Graphs in k-Path Irregular Graphs

It was proved in [1] that every graph can be embedded as an induced subgraph in a 2-path irregular graph. We now show that this result can be extended to k-path irregular graphs for all odd integers k. The basis for this result lies in the following proposition.

Proposition 5. Every r-regular graph of order n is an induced subgraph of a 3-path irregular graph of order $9n + r - 2$.

Proof: Let G be an r-regular graph of order n. The desired graph H has the vertex set

$$V(H) = V(G) \cup T \cup U \cup V \cup W,$$

where

$$T = \{t_i | 1 \leq i \leq n - 1\},$$

$$U = \{u_i | 1 \leq i \leq n + r - 1\},$$

$$V = \{v_i | 1 \leq i \leq 2n\} \text{ and}$$

$$W = \{w_i | 1 \leq i \leq 4n\}.$$

If $V(G) = \{x_1, x_2, \cdots, x_n\}$, then

$$\begin{aligned} E(H) = E(G) &\cup \{x_i t_j | 1 \leq i \leq n, 1 \leq j < i\} \cup \\ &\{t_i u_j | 1 \leq i \leq n - 1, 1 \leq j \leq n + r - 1\} \cup \\ &\{u_i v_j | 1 \leq i \leq n + r - 1, 1 \leq j \leq 2n\} \cup \\ &\{v_i w_j | 1 \leq i \leq n, 1 \leq j \leq 2n\} \cup \\ &\{v_i w_j | n+1 \leq i \leq 2n, 2n+1 \leq j \leq 4n\}. \end{aligned}$$

Note that in H,

$$\langle T \cup U \rangle \cong K_{n-1,n+r-1} \text{ and } \langle U \cup V \rangle \cong K_{n+r-1,2n},$$

while

$$\langle \{v_i | 1 \leq i \leq n\} \cup \{w_j | 1 \leq j \leq 2n\} \rangle \cong K_{n,2n}$$

and

$$\langle \{v_i | n+1 \leq i \leq 2n\} \cup \{w_j | 2n+1 \leq j \leq 4n\} \rangle \cong K_{n,2n}.$$

The degrees of the vertices of U, V and W in H are $3n-1$, $3n+r-1$ and n, respectively. Since $deg\ x_i = r + i - 1$ for $1 \leq i \leq n$ and

$deg\ t_j = n + r + j - 1$ for $1 \leq j \leq n - 1$, it follows that H is 3-path irregular.

The graph H constructed in the proof of Proposition 5 is also 1-path irregular. Further, by adding additional copies of the graph $K_{2n,2n}$ between V and W in the graph H, we may modify this proof to produce a k-path irregular graph for each odd integer $k \geq 3$.

Corollary 1. Let k be an odd positive integer. Every r-regular graph of order n is an induced subgraph of a k-path irregular graph.

In 1936 König [4] proved that every graph G is an induced subgraph of a regular graph H whose degree of regularity is the maximum degree of G. In 1963 Erdös and Kelly [2] determined the minimum order of such a graph H.

These facts give us the following result.

Corollary 2. Every graph of order n is an induced subgraph of an k-path irregular graph of order $O(n)$ for each odd positive integer n.

We conclude by presenting a problem.

Problem 3. Determine all even integers $k \geq 2$ such that every graph is an induced subgraph of a k-path irregular graph.

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