COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
52. COMBINATORICS, EGER (HUNGARY), 1987

NOTE

Isomorphic Subgraphs in a Graph

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At the combinatorics meeting of this Proceedings J. Schönheim posed the following problem: Is it true, that every graph of n edges has two (not necessarily induced) isomorphic edge disjoint subgraphs with say $\sqrt{n}$ edges? In the present note we answer this question in the affirmative. In fact we prove that every graph of n edges contains two isomorphic edge disjoint subgraphs with $cn^{2/3}$ edges and apart from the constant factor this result is best possible. Various generalizations are considered.

For any hypergraph $\mathcal{H}$, let $V(\mathcal{H})$ and $E(\mathcal{H})$ denote the set of vertices and the set of (hyper)edges of $\mathcal{H}$, respectively. $|E(\mathcal{H})|$ will be called the *size* of $\mathcal{H}$. Given any natural numbers $r, s \geq 2$, let $f_{r,s}(n)$ denote the maximum integer f such that in every r-uniform hypergraph $\mathcal{H}$ of size n one can find s pairwise edge-disjoint isomorphic subhypergraphs $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_s \subseteq \mathcal{H}$ of size f. We can summarize our results in the following.

Theorem. *(i) For every $s \geq 2$ there exist $c_s, d_s > 0$ such that*

$$c_s n^{s/(2s-1)} \leq f_{2,s}(n) \leq d_s n^{s/(2s-1)} \cdot \frac{\log n}{\log\log n}$$

(ii) For every $r \geq 3$, $s \geq 2$ there exist $c_{r,s}, d_{r,s} > 0$ such that

$$c_{r,s} n^{s/(rs-1)} \leq f_{r,s}(n) \leq d_{r,s} n^{s/(rs-r+1)} \cdot \frac{\log n}{\log\log n}$$

Proof. First we establish the upper bounds for all $r, s \geq 2$. Let us consider a random r-uniform hypergraph $\mathcal{H}$ with n edges and

$$v = n^{s/(rs-r+1)}$$

vertices. On this vertex set one can choose s isomorphic hypergraphs $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_s$ of size f in at most $\binom{\binom{v}{r}}{f}\frac{(v!)^{s-1}}{s!}$ different ways. Thus the probability that $\mathcal{H}$ contains s pairwise edge-disjoint isomorphic subhypergraphs of size f does not exceed

$$\binom{\binom{v}{r}}{f}\frac{(v!)^{s-1}}{s!}\cdot\frac{\binom{\binom{v}{r}-sf}{n-sf}}{\binom{\binom{v}{r}}{n}} \leq \frac{v^{rf}}{(\frac{f}{e})^f}v^{(s-1)v}\left(\frac{n}{\binom{v}{r}}\right)^{sf} \leq \left((er)^{rs}\,\frac{v}{f}\,v^{(s-1)v/f}\right)^f$$

Clearly, this number is smaller than 1, provided that $\frac{v}{f} \leq \varepsilon_{r,s}\frac{\log\log v}{\log v}$ for a suitable positive constant $\varepsilon_{r,s}$. In particular, for

$$f = \frac{1}{\varepsilon^{r,s}}n^{s/(rs-r+1)}\frac{\log n}{\log\log n},$$

with positive probability $\mathcal{H}$ does not have s edge-disjoint isomorphic subhypergraphs of size f.

To prove the lower bound in (i), we shall need a simple observation. A *star* of a graph G is a nonempty collection of edges incident to the same vertex. A graph is called a star-system, if all of its connected components are stars.

Lemma. Let G^* be a star-system on $v \geq 32(s-1)^3$ vertices. If G^* does not contain s pairwise edge-disjoint isomorphic subgraphs of size f, then

$$v \leq 4s(f-1).$$

Proof of the Lemma. First we show that if G^* is any star-system on v vertices then, apart from at most $\sqrt{2(s-1)^3 v}$ edges, $E(G^*)$ can be partitioned into s isomorphic classes.

If G^* contains s components of the same size, then the assertion follows by induction on the number of vertices. Otherwise, denoting by t the number of components of G^*, we have

$$v - t = |E(G^*)| \geq (s-1)\left(1+2+\ldots+\left[\frac{t}{s-1}\right]\right) \geq \frac{t}{2}\left[\frac{t}{s-1}\right].$$

Therefore

$$t \leq \sqrt{2v(s-1)}.$$

Since, apart from at most $s-1$ edges, each component can be divided into s stars of the same size, the number of "exceptional" edges is at most $(s-1)t \leq \sqrt{2(s-1)^3 v}$, and the assertion follows.

Assume now that $v \geq 32(s-1)^3$. Then $\sqrt{2(s-1)^3 v} \leq \frac{v}{4}$, i.e., G^* has at most $\frac{v}{4}$ "exceptional" edges. Hence the number of edges which occur in a given class of the partition is at least $(|E(G^*)| - [v/4])/s \geq v/4s$. Using the fact that each class is of size at most $f-1$, we obtain the Lemma. ■

We turn to the proof of the lower bound in (i). Let G be a graph with n edges and v non-isolated vertices, and let f be a natural number. Let us partition $E(G)$ into s as equal parts as possible: $E(G) = E_1 \cup E_2 \cup \ldots \cup E_s$, $|E_i| \geq [n/s]$ for every i.

If there exist $s-1$ permutations of the vertex set, $\pi_1, \pi_2, \ldots, \pi_{s-1}$ such that

$$|\pi_1 E_1 \cap \pi_2 E_2 \cap \ldots \cap \pi_{s-1} E_{s-1} \cap E_s| \geq f$$

then G obviously contains s pairwise edge-disjoint isomorphic subgraphs of size f. Otherwise, the average size of $\pi_1 E_1 \cap \ldots \cap \pi_{s-1} E_{s-1} \cap E_s$ over all choices of $\pi_1, \ldots, \pi_{s-1}$ is at most $f-1$, i.e.,

$$f - 1 \geq \frac{1}{(v!)^{s-1}} \left[\frac{n}{s}\right]^s (2(v-2)!)^{s-1} \geq \left[\frac{n}{s}\right]^s \left(\frac{2}{v^2}\right)^{s-1}.$$

Let $G^* \subseteq G$ be any star-system spanning all non-isolated vertices of G. If G does not contain s pairwise edge-disjoint isomorphic subgraphs of size f, then the same is true for G^*. Hence we can apply the Lemma to deduce

$$v \leq 4s(f-1).$$

Combining the last two inequalities we obtain that, if G does not have s pairwise edge-disjoint isomorphic subgraphs of size f, then

$$f - 1 \geq \left[\tfrac{n}{s}\right]^s \left(\tfrac{1}{8s^2(f-1)^2}\right)^{s-1},$$

$$f - 1 > \left[\frac{n}{s}\right]^{s/(2s-1)} \cdot \frac{1}{3s}.$$

This proves the lower bound in (i).

The (weak) lower bounds in (ii) can be established by induction on r. Let $\mathcal{H}$ be an r-uniform hypergraph with n edges. If there is a vertex $x \in V(\mathcal{H})$ of degree m, then we can find s edge-disjoint isomorphic subhypergraphs of size $f_{r-1,s}(m)$ among the edges of $\mathcal{H}$ containing x. Otherwise, we can choose at least n/rm pairwise disjoint edges, hence

$$f_{r,s}(n) \geq \min_m \max\left\{f_{r-1,s}(m), \left[\frac{n}{rsm}\right]\right\}$$

and the result follows by easy calculation. ■

It would be interesting to improve the bounds for hypergraphs. The first unsolved problem is the following: Is it true that every 3-hypergraph of n edges contains two edge disjoint subgraphs with $c\sqrt{n}$ edges?

Note added in proof: Similar results have benn obtained by the authors I. Krasikov and N. Alon and the authors R. Gould and V. Rödl.

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