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Intersection Graphs for Families of Balls in $\mathbf{R}^n$

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1. Introduction

If F is a finite family of sets, then the intersection graph $\Gamma(F)$ is the graph with vertex-set F and edges the unordered pairs C, D of distinct elements of F such that $C \cap D \neq \emptyset$. It is easy to see [6, p. 19] that every graph G is isomorphic to some intersection graph $\Gamma(F)$.

Some interesting classes of graphs have arisen by letting F range over families of balls in some metric space, such as arcs on a circle or intervals of the real line [4], or cubes, boxes or spherical balls in n-space [9, 11, 12].

For the case of balls in $\mathbf{R}^n$ with the Euclidean norm, Guttman [5] and Havel [7] have defined the *sphericity*, $\mathrm{sph}(G)$, of a graph G to be the least dimension n in which G is isomorphic to $\Gamma(F)$ for F some family of open (equivalently, closed) balls of radius 1; and Maehara [11] has defined the *contact dimension*, $\mathrm{cd}(G)$, to be the least n for which G is isomorphic to $\Gamma(F)$ for F some family of closed balls of radius 1 such that no pair of balls intersects in more than one point. Maehara has shown that $\mathrm{sph}(G) \leqslant \mathrm{cd}(G) \leqslant |V(G)| - 1$ for all graphs G, and has studied $\mathrm{sph}(G)$ and $\mathrm{cd}(G)$ as functions of the structure of G [9–11].

Roberts [12] has defined the *cubicity*, $\mathrm{cub}(G)$, of G to be the least n for which G is isomorphic to $\Gamma(F)$ for F some family of unit cubes with edges parallel to the Cartesian co-ordinate axes in $\mathbf{R}^n$. Such cubes can be viewed as balls with respect to a different norm on $\mathbf{R}^n$, and the question arises as to how the shape of the unit ball in an n-dimensional normed linear space is related to the least n in which G can be represented by an appropriate $\Gamma(F)$. Havel [7] has shown that there are graphs of sphericity 2 but with arbitrarily large cubicity; Fishburn [3] has shown that there are graphs G of cubicity 2 or 3 for which $\mathrm{sph}(G) > \mathrm{cub}(G)$, but remarks that it is unknown whether $\mathrm{sph}(H) > \mathrm{cub}(H)$ can hold for graphs H of arbitrarily large cubicity.

In this paper, we are concerned with a different type of problem. Let Γ_n be the set of all graphs $\Gamma(F)$, where F is a family of balls of arbitrary radii in $\mathbf{R}^n$ in the Euclidean norm (where we allow both open balls and closed balls to be in F, since the distinction here will be unimportant). We are interested in what happens if none of the balls in F is allowed to penetrate too far into another ball of F. That is, we relax the notion of 'contact dimension' to allow more contact than a single point (and to allow arbitrary radii), but we shall restrict the amount of contact between any two balls in F. For $0 < \varepsilon \leqslant 1$, let $\Gamma_{n,\varepsilon}$ be the set of all graphs $\Gamma(F)$ in Γ_n such that no ball in F contains more than the fixed proportion $(1 - \varepsilon)$ of the voluem of another ball in F. We shall see that the graphs in $\Gamma_{n,\varepsilon}$ have bounded chromatic numbers, which seems somewhat surprising for small ε.

Let $B(x, r)$ denote a ball (either open or closed), of radius $r > 0$ and center x, in the Euclidean space $\mathbf{R}^n$. Let $B^o(x, r) = \{y : \|x - y\| < r\}$, and $B^c(x, r) = \{y : \|x - y\| \leqslant r\}$, be the corresponding open, and closed balls. Let $\mu[A]$ be the n-dimensional Lebesgue volume of the subset A of $\mathbf{R}^n$. Henceforth, ε always denotes a real number in $(0, 1]$. A pair of balls B, B' are ε-disjoint if $\mu(B \cap B') \leqslant (1 - \varepsilon) \min\{\mu(B), \mu(B')\}$. If two balls B, B'

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are not ε-disjoint, then we say they are $(1 - \varepsilon)$-friendly, since in this case they overlap on more than a proportion of $(1 - \varepsilon)$ of the volume of the smaller ball. A family F of balls is ε-disjoint whenever every pair of balls in F is ε-disjoint. Thus $\mathbf{\Gamma}_{n,\varepsilon}$ is the set of all intersection graphs $\mathbf{\Gamma}(F)$ for ε-disjoint families F of balls in $\mathbf{R}^n$.

Note that as ε tends to 1, the 'disjointness' of ε-disjoint balls B, B' increases, and the 1-disjoint balls are either disjoint or else they intersect in a single point. In particular, $\mathbf{\Gamma}_{n,1}$ is the set of graphs G such that $\text{cd}(G) \leqslant n$, and so contains all G with $|V(G)| \leqslant n + 1$, by [11].

Clearly, $0 < \varepsilon \leqslant \varepsilon' \leqslant 1$ implies $\mathbf{\Gamma}_{n,1} \subseteq \mathbf{\Gamma}_{n,\varepsilon'} \subseteq \mathbf{\Gamma}_{n,\varepsilon} \subseteq \mathbf{\Gamma}_n$. Let $\chi(G)$ denote the chromatic number of graph G, $N(v)$ denote the set of neighbors of vertex v in G, and $\langle A \rangle$ denote the subgraph induced by the subset A of vertices of G. For terms not defined here, consult [6]. We summarize our results as follows:

THEOREM 1. *There exists a least integer $d = d(n, \varepsilon)$ such that every graph in $\mathbf{\Gamma}_{n,\varepsilon}$ has a vertex of degree at most d.*

COROLLARY 2. $\text{Max}\,\{\chi(G) : G \in \mathbf{\Gamma}_{n,\varepsilon}\} \leqslant d(n, \varepsilon) + 1$.

THEOREM 3. *There exists a least integer $m = m(n)$ such that every graph G in $\mathbf{\Gamma}_n$ has a vertex v for which $\langle N(v) \rangle$ contains no independent set of size greater than m.*

COROLLARY 4. *The complete bipartite graph $K_{p,p} \notin \mathbf{\Gamma}_n$ for all $p \geqslant m(n)$.*

We remark that Janos Pach of the Mathematical Institute, Budapest, has independently proved, but not published, the result of our Corollary 2.

2. Preliminary Lemmas

LEMMA A. *If X is a compact metric space and $\delta > 0$, then there is a least integer $N = N(X, \delta)$ such that X contains no more than N points which are pairwise at least δ-distance from each other.*

Lemma A is well known. We omit its easy proof. Henceforth, we let $0 < \varepsilon < 1$ and $\lambda = (1 - \varepsilon)^{1/n}$. Note that $0 < \lambda < 1$.

LEMMA B. *$B(y, r)$ and $B(z, s)$ are $(1 - \varepsilon)$-friendly if either*
(1) $s - \|y - z\| > \lambda r$ or (2) $0 < r \leqslant s$ and $\|y - z\| < r(1 - \lambda)$.

PROOF. Suppose that $s - \|y - z\| > \lambda r$. We may choose τ such that $\lambda r < \tau < r$ and $\tau < s - \|y - z\|$. Then $B^o(z, s) \supset B(y, \tau)$, so that $\mu[B(y, r) \cap B(z, s)] \geqslant \mu[B(y, r) \cap B(y, \tau)] = \mu[B(y, \tau)] > \mu[B(y, r\lambda)] = \lambda^n \mu[B(y, r)] = (1 - \varepsilon)\mu[B(y, r)]$. Therefore $B(y, r)$ and $B(z, s)$ are $(1 - \varepsilon)$-friendly. If $0 < r \leqslant s$ and $\|y - z\| < r(1 - \lambda)$, then $\|y - z\| < r - r\lambda \leqslant s - r\lambda$, so that $r\lambda < s - \|y - z\|$, and the previous case applies. □

Let $\Sigma(x, \theta)$ denote any closed sector in the plane between two rays with vertex x and angle θ, and for $d > 0$ let $\Sigma(x, \theta, d) = \{w \in \Sigma(x, \theta) : \|w - x\| \geqslant d\}$.

LEMMA C. *There exist $d > 0$ and an acute angle $\theta > 0$ such that if $r, s > 0$ and $y, z \in \mathbf{R}^2$, and r, s, y, z satisfy all of the conditions*

(i) $\|x - y\| \leqslant \|x - z\|$ *and* $y, z \in \Sigma(x, \theta, d)$,
(ii) $B(x, 1) \cap B(y, r) \neq \varnothing \neq B(x, 1) \cap B(z, s)$, *and*
(iii) $B(x, 1) \nsubseteq B(y, r)$
then $s - \|y - z\| > \lambda r$.

PROOF. Choose θ such that $0 < \theta < \pi/4$ and $\cos\theta - \sin\theta > \lambda$. (This is possible since $\cos\theta - \sin\theta$ increases to 1 as θ decreases to 0.) Let $d = (1 + \lambda)/(\cos\theta - \sin\theta - \lambda)$. Note that $0 < \cos\theta - \sin\theta - \lambda < 1 - \lambda$, so that $d > (1 + \lambda)/(1 - \lambda) > 1 + \lambda$ and $(d - 1)/(d + 1) > \lambda$.

Suppose that $r, s > 0$ and $y, z \in \mathbf{R}^2$ and r, s, y, z satisfy (i), (ii) and (iii). Let $a = \|x - y\|$ and $b = \|y - z\|$. From (i), $\|x - z\| \geqslant a \geqslant d$. From (ii), $r \geqslant a - 1$ and $s \geqslant \|x - z\| - 1$. From (iii), $a + 1 \geqslant r$.

If the line M through x and y contains z, then $s - \|y - z\| \geqslant \|x - z\| - 1 - \|y - z\| = \|x - z\| - \|y - z\| - 1 = \|x - y\| - 1 = a - 1 > \lambda(a + 1) \geqslant \lambda r$. This holds because (i) implies that $\|x - z\| = \|x - y\| + \|y - z\|$, and because $a \geqslant d$ implies that $(a - 1)/(a + 1) \geqslant (d - 1)/d + 1) > \lambda > 0$, where $a + 1 \geqslant r$. (Here we use that $(x - 1)/(x + 1)$ is increasing for $x \geqslant 0$.) Thus we obtain $s - \|y - z\| > \lambda r$ in this case.

Now suppose that z does not lie on M, and let L be the line through x and z, and let T be the triangle with vertices x, y, z (see Figure 1).

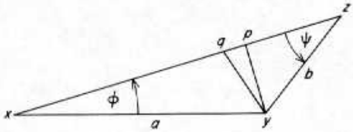

FIGURE 1

We have $\phi \leqslant \theta$, since $y, z \in \Sigma(x, \theta)$. Let p be the foot of the perpendicular from y to L. Then $a \sin\phi = b \sin\psi = \|y - p\|$. Let q be the point on segment xz of L such that $\|q - z\| = b$. Our conditions imply that $\phi <$ angle xyz, so that $b < \|x - z\|$. Then $\|q - p\| = b - b\cos\psi$, so that $\|q - x\| = a\cos\phi - \|q - p\| = a\cos\phi + b\cos\psi - b$. Now $s - b = s - \|q - z\| \geqslant (\|x - z\| - 1) - \|q - z\| = (\|x - z\| - \|q - z\|) - 1 = \|q - x\| - 1 = a\cos\phi + b\cos\psi - b - 1 = a\cos\phi + (b^2 - b^2\sin^2\psi)^{1/2} - b - 1 = a\cos\phi + [(b^2 - a^2\sin^2\phi)^{1/2} - b] - 1 = a\cos\phi - 1 - (a^2\sin^2\phi)/[b + (b^2 - a^2\sin^2\phi)^{1/2}]$.

We must have $0 < \psi < \pi/2$, because if $\psi \geqslant \pi/2$ then the side a of the triangle T would be the (unique) largest side of T, violating the conditiong that $\|x - z\| \geqslant a = \|x - y\|$. Now $0 < a\sin\phi = b\sin\psi < b < b + b\cos\psi = b + (b^2 - b^2\sin^2\psi)^{1/2} = b + (b^2 - a^2\sin^2\phi)^{1/2}$, so that $0 < (a\sin\phi)/[b + (b^2 - a^2\sin^2\phi)^{1/2}] < 1$. Therefore $0 < (a^2\sin^2\phi)/[b + (b^2 - a^2\sin^2\phi)^{1/2}] < a\sin\phi$. Applying this to our previous inequality for $s - b$, we obtain $s - b > a\cos\phi - 1 - a\sin\phi$. But $g(\gamma) = \cos\gamma - \sin\gamma$ is decreasing for $0 \leqslant \gamma \leqslant \pi/2$, and we have that $0 < \phi \leqslant \theta$, therefore $0 < \cos\theta - \sin\theta - \lambda \leqslant \cos\phi - \sin\phi - \lambda$. Also, $a \geqslant d = (1 + \lambda)/(\cos\theta - \sin\theta - \lambda) \geqslant (1 + \lambda)/(\cos\phi - \sin\phi - \lambda)$; thus $s - b > a(\cos\phi - \sin\phi) - 1 \geqslant \lambda(a + 1) \geqslant \lambda r$. Therefore $s - \|y - z\| > \lambda r$. □

LEMMA D. *Let $B_0 = B^c(\mathbf{0}, 1)$ be the closed unit ball at the origin in $\mathbf{R}^n$. There exists an integer $k = k(n, \varepsilon)$, depending only on n and ε, such that if $h > k$ and $\{B(x_i, r_i) : 1 \leqslant i \leqslant h\}$ is any family of distinct balls of radii $r_i \geqslant 1$, each of which intersects B_0, then there are distinct indices $p, q \in \{1, \ldots, h\}$ such that either (i) $B_0 \subseteq B(x_p, r_p)$ or (ii) $B_0 \subseteq B(x_q, r_q)$ or (iii) $B(x_p, r_p)$ and $B(x_q, r_q)$ are $(1 - \varepsilon)$-friendly.*

Proof. Let d, θ be as in Lemma C. Let S be the unit sphere $\{x \in \mathbf{R}^n : \|x\| = 1\}$. Let $\varrho = 2 \sin(\theta/4)$. The covering $\{B^o(x, \varrho) : x \in S\}$ of S has a finite subcover $\{B^o(y_i, \varrho) : 1 \leqslant i \leqslant m\}$, where m is chosen to be least possible and clearly depends only on n and θ, that is, on n and ε. (In fact, it is easy to see that $m \leqslant N(S, \varrho)$, where N is from Lemma A.) For $1 \leqslant i \leqslant m$, let C_i be the cone $\{x \in \mathbf{R}^n$: there exists some $y \in B^0(y_i, \varrho)$ such that x lies on the ray from $\mathbf{0}$ through $y\}$. Let $D = B^c(\mathbf{0}, d)$ and let $C_i^* = \{x \in C_i : \|x\| \geqslant d\}$, for each i, $1 \leqslant i \leqslant m$. Let $\delta = 1 - \lambda = (1 - \varepsilon)^{1/n}$, and let $k = m + N(D, \delta)$.

Now suppose that $h > k$ and that $\{B(x_i, r_i) : 1 \leqslant i \leqslant h\}$ is a family of distinct balls of radii $r_i \geqslant 1$ such that each $B(x_i, r_i) \cap B_0 \neq \varnothing$. Suppose that for every pair p, q of distinct elements of $\{1, \ldots, h\}$, none of the conditions (i), (ii) and (iii) holds. At most $N(D, \delta)$ indices i have $x_i \in D$; otherwise, by Lemma A, we obtain some $\|x_p - x_q\| < 1 - \lambda \leqslant r_p(1 - \lambda)$ and, by Lemma B(2), (iii) would hold. Clearly, $C_1^* \cup \ldots \cup C_m^* \cup D = \mathbf{R}^n$, so by the pigeonhole principle at least two distinct indices p, q in $\{1, \ldots, h\}$ have both x_p, x_q lying in the same C_j^*, for some j. Assume that $n > 1$. (The case $n = 1$ is easy, and in fact follows from the case $n = 2$.) The points x_p, x_q and the origin $\mathbf{0}$ lie in a plane P, which we identify with $\mathbf{R}^2$, and the closure of $P \cap C_j$ is a sector $\Sigma(\mathbf{0}, \theta)$ because of our choice of ϱ. Without loss of generality, we may assume that $\|x_p\| \leqslant \|x_q\|$. Then $x = \mathbf{0}$, $y = x_p$, $z = x_q$, $r = r_p$, and $s = r_q$ satisfy the hypotheses of Lemma C. Therefore $r_q - \|x_p - x_q\| > \lambda r_p$. But then by Lemma B, we have that condition (iii) holds. This is a contradiction. We conclude that at least one of (i), (ii) and (iii) must hold. □

3. Proof of Theorem 1

If $\Gamma = \Gamma(F) \in \Gamma_{n,\varepsilon}$ and $B(x, r)$ is a ball of F of least radius r, then we may replace each ball $B(y, s)$ of F by the ball $B(y - x, s/r)$, thereby obtaining a new family F' for which $\Gamma(F') \simeq \Gamma$, and where $\Gamma(F') \in \Gamma_{n,\varepsilon}$ and has $B(\mathbf{0}, 1)$ as a ball of least radius in F'. This is because the similarity transformation $T(y) = (y - x)/r$ preserves proportions of n-dimensional Lebesgue volumes. Therefore we may assume without loss of generality that $B_0 = B(\mathbf{0}, 1) \in F$ is a ball of least radius in F.

Let $\{B(x_i, r_i) : 1 \leqslant i \leqslant h\}$ be the balls in F which are neighbors of B_0 in Γ. For all i such that $1 \leqslant i \leqslant h$, $B_0^c \cap B(x_i, r_i) \supseteq B_0 \cap B(x_i, r_i) \neq \varnothing$. Also, $B_0^c \nsubseteq B(x_i, r_i)$ for every i, since otherwise for some i, $\mu[B_0 \cap B(x_i, r_i)] = \mu[B_0] > (1 - \varepsilon)\mu[B_0]$, which would contradict the ε-disjointness of the balls B_0 and $B(x_i, r_i)$. Further, every $r_i \geqslant 1$, since B_0 has least radius in F. Now $h \leqslant k(n, \varepsilon)$ follows from Lemma D and the hypothesis that F is an ε-disjoint family via $\Gamma(F) \in \Gamma_{n,\varepsilon}$. □

Remark. Corollary 2 is an immediate consequence of Theorem 1, by an old argument of Dirac [1]; namely, assuming that all graphs in $\Gamma_{n,\varepsilon}$ with fewer vertices than Γ can be colored with $d(n, \varepsilon) + 1$ or fewer colors, delete a vertex of degree $\leqslant d(n, \varepsilon)$ from Γ and color the vertices of the resulting graph. Then at least one color will be available for the deleted vertex when it is restored to Γ. This shows recursively how to color properly the vertices of any $G \in \Gamma_{n,\varepsilon}$ with $d(n, \varepsilon) + 1$ or fewer colors.

Corollary 2 has an interesting geometrical interpretation: there is a least integer $c = c(n, \varepsilon)$ such that every finite family F of balls in $\mathbf{R}^n$ of arbitrary radii, such that none of the balls contains more than the fraction $(1 - \varepsilon)$ of the volume of another, can be partitioned into at most c subfamilies in each of which the balls are pairwise disjoint. This generalizes a result of two of the authors [8], which was inspired by related results in [2].

4. Proof of Theorem 3

Let $m = k(n, \frac{1}{2})$, where the positive integer k comes from Lemma D by taking $\varepsilon = \frac{1}{2}$. Let $\Gamma \in \Gamma_n$. Say $\Gamma = \Gamma(F)$. By the same argument used in Section 3, we may assume that

$B_0 = B(\mathbf{0}, 1)$ is a ball of least radius in F. Let $\{B(x_i, r_i) : 1 \leqslant i \leqslant h\}$ be a maximum independent set of vertices amongst the neighbors of B_0 in Γ. We claim that $h \leqslant m$. This is clear if $h = 1$. Suppose that $h > 1$. Now $B_0 \cap B(x_i, r_i) \neq \varnothing$ for each i, $1 \leqslant i \leqslant h$, but $1 \leqslant p < q \leqslant h$ implies that $B(x_p, r_p)$, and $B(x_q, r_q)$ are disjoint, and hence they are certainly not $\frac{1}{2}$-friendly. Also, $B_0 \nsubseteq B(x_i, r_i)$ for every i; indeed, if $B_0 \subseteq B(x_p, r_p)$ for some p, then choosing $q \neq p$ we would obtain $\varnothing = B(x_p, r_p) \cap B(x_q, r_q) \supseteq B_0 \cap B(x_q, r_q) \neq \varnothing$, a contradiction. By Lemma D, we conclude that $h \leqslant m$. □

We note that Corollary 4 is an immediate consequence of Theorem 3.

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