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Extremal Problems for Degree Sequences

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1. Introduction

If positive integer weights are assigned to the edges of a graph G, then the degree of a vertex is the sum of the weights of the edges that are incident to the vertex. A graph G with weighted edges is said to be *irregular* if the degrees of the vertices are distinct, and the *irregularity strength* of the graph G is the smallest s such that the edges can be weighted with $\{1, 2, \ldots, s\}$ and be irregular. These notions are defined in [1].

No graph can have irregularity strength 1, since it is not possible for all of the degrees to be distinct in a simple (no weights on the edges) graph. Several measures can be used to determine how close a graph is to being irregular. For example, the number of duplicated degrees, the sum of the duplicated degrees, or even the location of the duplicated degrees in the degree sequence are possibilities. Also, the number, sum, or location of distinct (or not duplicated) degrees could be considered. We will consider various combinations of these measures and determine (or give bounds) for their extremal values.

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2. Notation

Let G be a graph of order n with degree sequence

$$(d_1 \leq d_2 \leq \ldots \leq d_n).$$

We will always assume that $\{v_1, v_2, \ldots, v_n\}$ is the vertex set with degrees in the increasing order just given unless explicitly stated otherwise. For any such degree sequence the index set $\{1, 2, \ldots, n\}$ is partitioned into two sets $D = D(G)$ (duplicated degrees) and $S = S(G)$ (single degrees). Thus

$$D = \{i : d_i = d_j \text{ for some } i \neq j\},$$

and S is the remaining set of indices that are associated with degrees that appear precisely once in the sequence. Each index in D is associated with some duplicated degree, and if we choose the first index associated with each duplicated degree, we obtain a proper subset $D' = D'(G)$ of D. Also, related to the degree sequence is the set $\mathcal{M} = \mathcal{M}(G)$ (missing degrees), which is

$$\mathcal{M} = \{j : 0 \leq j \leq n-1 \text{ and } j \neq d_i \text{ for any } i\}.$$

Note that the set $\mathcal{M}$ is not a subset of the index set, but a collection of possible values of degrees on the index set. From this point on we will identify with each graph G the sets $D, D', \mathcal{M}$ and S (not identifying the graph G unless it is necessary to avoid confusion). Also, by ΣD, $\Sigma D'$ and ΣS we will mean the sum of the degrees indexed by each of the sets, and by $\Sigma \mathcal{M}$ the sum of the elements in $\mathcal{M}$.

Let H_n denote the graph of order n with vertex set $V(H_n)$ and edge set $E(H_n)$ given by:

$$V(H_n) = \{v_1, v_2, \ldots, v_n\} \text{ and}$$
$$E(H_n) = \{v_i v_j : i + j > n \text{ and } i \neq j\}.$$

This graph, called the half graph, has degree sequence

$$d_i = \begin{cases} i & i \leq n/2, \\ i-1 & i > n/2, \end{cases}$$

and will be used to construct several examples of graphs that have extremal degree sequences. Note that for the graph H_n, $\mathcal{M} = \{0\}$, $D' = \{\lfloor n/2 \rfloor\}$, $D = \{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1\}$ with the one duplicated degree being $\lfloor n/2 \rfloor$, $S = \{1, 2, \ldots, n\} - D$, and $\Sigma D = 2\lfloor n/2 \rfloor$.

For every graph G the set D is nonempty, since there are only n possible degrees for vertices of a graph of order n, and it is not possible for 0 and $n-1$ to be simultaneously in a degree sequence. For the half graph H_n, D has precisely two elements and $\mathcal{M}$ has 1 element. In each case these are clearly the minimal possible, so some of the extremal problems are trivial. However, not all of the extremal problems are so simple.

One of the results that will be useful is a classical condition of Erdős and Gallai which gives a characterization of *graphical degree sequences.*

Theorem 1. [2] *A sequence of positive integers $d_1 \leq d_2 \leq \ldots \leq d_n$ is graphical if and only if $\sum_{i=1}^{n} d_i$ is even, and for each $k(1 \leq k \leq n-1)$,*

$$\sum_{i=1}^{k} d_{n-i+1} - k(k-1) \leq \sum_{i=1}^{n-k} \min\{k, d_i\}.$$

The left hand side of the previous inequality is a lower bound on the number of edges between the k vertices of highest degree and the remaining vertices of the graph, and the right hand side is an upper bound on this number of edges. It is this comparison on the count of the number of edges that will be used most frequently. In fact, generally we will not use the statement of Theorem 1 explicitly, but we use the argument that verifies the only if part of the Theorem.

Another well known useful result concerning graphical degree sequences is due to Havel [4] and Hakimi [3].

Theorem 2. ([3],[4]) *A sequence $d_1 \leq d_2 \leq \ldots \leq d_n \neq 0$ is graphical if and only if the (rearranged) sequence $d_1, d_2, \ldots, d_{n-d_n-1}, d_{n-d_n}-1, \ldots, d_{n-1}-1$ is graphical.*

3. Location of duplicated degrees

In the half graph H_n the indices in D are in the middle of the degree sequence. Is it possible in a degree sequence that all of the indices in D are small (or large), and if so, how small (or large)? The following example indicates that in a graph of order n all of the duplicated degrees can be in the first approximately $\sqrt{n}$ terms of the degree sequence.

Let m be a fixed positive integer, and n an integer satisfying

$$m^2 - m + \delta < n \leq m^2 + m + \delta,$$

where $\delta = 0$ for n even, and $\delta = 1$ for n odd. Let $\{v_2, v_3, \ldots, v_n\}$ be the vertices of the half graph H_{n-1}, and add to this graph an isolated vertex v_1 to form the graph H'_n. Let $A = \{v_1, v_2, \ldots, v_m\}$. Form a new graph L_n from H'_n by adding for each $i \geq (n+1)/2$ an edge between v_i and a vertex in A in such a way that the new degrees of the vertices in A differ by at most 1 (in fact are either $m-1$ or m). This can be done, since if B is the sum of the degrees in L_n of the vertices in A, then

$$m(m-1) < B = \binom{m}{2} + \lfloor n/2 \rfloor \leq m^2.$$

The graph L_n has degree sequence

$$d_i = \begin{cases} m-1 \text{ or m} & i \leq m \\ i-1 & i > m. \end{cases}$$

Therefore, all degrees are distinct except for the first $m+1$ terms, and

$$(\sqrt{4(n-\delta)+1}+1)/2 \leq m+1 < (\sqrt{4(n-\delta)+1}+3)/2.$$

Thus, all indices of D are in the first approximately $\sqrt{n}$ terms of the degree sequence. This example indicates that the following theorem is the best possible.

Theorem 3. *Let G be a graph of order n with degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n$. If $d_i \in S$ for all $i > k$, then*

$$k \geq (\sqrt{4(n-\delta)+1}+1)/2,$$

where $\delta = 0$ for n even, and $\delta = 1$ for n odd.

Proof. First consider the case when $n = 2p$ is even. We will assume that $D \subseteq \{1, 2, \ldots, k\}$, and show that k satisfies the inequality in the statement of the theorem. Let A be the vertices associated with the first p terms in the degree sequence of G, and B the remaining vertices. We will count the number of edges between A and B using the proof technique of Theorem 1 to obtain the inequality for k.

Let $t = d_{p+1} \leq p$. By assumption $d_{p+1} < d_{p+2} < \ldots < d_{2p}$, so $d_{p+i} \geq t+i-1$. Hence, for $i > p-t$, v_{p+i} is adjacent to at least $t-p+i$ vertices of A. This implies that there are at least $1+2+\ldots+t = \binom{t+1}{2}$ edges between B and A.

On the other hand, the degree sequence of the vertices in A are all distinct except for possibly the first k terms, and the largest degree is at most $t-1$. Thus the sum of the degrees of the vertices in A is at most

$$(t-1)+(t-2)+\ldots+(t-p+k)+k(t-p+k-1) = p(2t-p-1)/2 + \binom{k}{2},$$

which gives an upper bound on the number of edges between A and B. The following inequality results:

$$\binom{k}{2} \geq \binom{t+1}{2} - p(2t-p-1)/2.$$

Considering the right hand side of the above inequality as a function of t it is easy to verify that its minimum integer value is p and occurs when $t = p$. Thus, we have the inequality,

$$k^2 - k \geq 2p = n,$$

which is immediately the desired result for n even.

For $n = 2p + 1$ odd, the same argument used in the even case yields the inequality $k^2 - k \geq 2p = n - 1$. This gives the desired inequality and completes the proof of Theorem 3. ■

Remark. The complement of the graph L_n is one that has its D contained in the last approximately $\sqrt{n}$ terms of the degree sequence. Also, this is the best possible by the argument just given.

4. Sums of duplicated and missing degrees

Before stating any results in this section some specialized notation will be given and families of graphs with special degree sequences will be described. For nonnegative integers $p \leq q \leq r$ and positive integer k, consider the sequence of $n = r - p + k$ numbers

$$(p, p+1, \ldots, q-1, q, q, \ldots q, q+1, q+2, \ldots, r)$$

with the term q occurring k times in the sequence. The family of graphs of order n with this degree sequence will be denoted by $\mathcal{H}(p, q, r, k)$. In particular, the half graph H_n is in $\mathcal{H}(1, \lfloor n/2 \rfloor, n-1, 2)$, and at the other extreme a q-regular graph is in $\mathcal{H}(q, q, q, n)$.

There are families of graphs related to the half graph H_n that can be derived from H_n. We will describe two such families that will be used to show the sharpness of the bounds in the theorems that will follow. The first will be described here, and the second later in this section.

For the first family, let $k \geq 2$ be a fixed integer and m be an integer divisible by $k-1$. Consider the half graph H_{2m} with vertices $\{v_1, v_2, \ldots, v_{2m}\}$. The vertex v_m is adjacent to the m vertices $T = \{v_{m+1}, v_{m+2}, \ldots, v_{2m}\}$. A graph of order $n = 2m + k - 1$ can be obtained from H_{2m} by adding $k-1$ independent vertices $\{u_1, u_2, \ldots, u_{k-1}\}$ such that each u_i is adjacent to $m/(k-1)$ vertices of T, each vertex of T is adjacent to just one of the u_i's, and the degrees of the remaining vertices of H_{2m} are left unchanged. Since this graph has k vertices of degree $m/(k-1) = (n-k+1)/(2k-2)$, it is in $\mathcal{H}(1, (n-k+1)/(2k-2), n-k+1, k)$, so we will denote it by

$$H(1, (n-k+1)/(2k-2), n-k+1, k).$$

Clearly for this graph

$$\Sigma D' = (n-k+1)/(2k-2) \text{ and}$$

$$\Sigma D = k(n-k+1)/(2k-2).$$

Note that if $k = 2$, then the half graph H_n is obtained and $\Sigma D' = (n-1)/2$ and $\Sigma D = n-1$. At the other extreme, if $m = k-1$ (i.e. $n = 3k-3$), then $\Sigma D' = 1$ and $\Sigma D = k = (n+3)/3$ for the graph $H_n(1,1,2n/3,(n+3)/3)$. This is, in fact, the smallest possible value of ΣD, as the following result indicates.

Theorem 4. *If G is a graph of order n without isolated vertices, then $\Sigma D' \geq 1$ and $\Sigma D \geq (n+3)/3$. In addition, both bounds are sharp.*

Proof. The graph $H_n(1,1,2n/3,(n+3)/3)$ just described verifies that each inequality could not be improved. Also, clearly $\Sigma D' \geq 1$, since any degree sequence must have a duplication.

For the second inequality consider the degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n$ of G and assume that $\Sigma D < (n+3)/3$. Select the largest index j such that $d_j < n/3$. Such an index exists, in fact, with $j < 2n/3$; for if not, then there would be $j+1-n/3 \geq (n+3)/3$ duplicated degrees preceding d_j. Also, $j \geq n/3$, for if not, there would be duplicated degrees after d_j which are greater than $n/3$. Let $A = \{v_1, v_2 \ldots, v_j\}$ and $B = \{v_{j+1}, v_{j+2}, \ldots, v_n\}$.

Select the integer t such that $t < n/3$, but $t+1 \geq n/3$. Observe that $d_j \leq t$, and also $d_{j+1} < d_{j+2} < \ldots < d_n$, $d_{j+1} \geq t+i$ for $i \geq 1$, and v_{j+1} is adjacent to at least $t-n+i+j+1$ vertices of A. In particular, v_n is adjacent to at least $t+1$ vertices of A, and so there are at least $1+2+\ldots+(t+1)$ edges from B and A. However, the number of edges from A to B is no more than the sum of the degrees of the vertices in A, and so is at most $\Sigma D + 2 + 3 + \ldots + t$ (which can be the case when 1 is the only duplicated degree.). Therefore, $\Sigma D \geq t+2 \geq (n+3)/3$. This contradiction completes the proof of Theorem 4. ∎

If a bound is placed on the number of duplicated degrees or the minimum degree in the degree sequence, then more can be said about the sum of the duplicated degrees. The following two theorems are examples of this type of result.

Theorem 5. *Let k be a fixed positive integer. If G is a graph of sufficiently large order n with no isolated vertices such that the number of duplicated degrees is k (i.e., $|D| = k$), then*

$$\Sigma D' \geq (n-2k+2)/(2k-2), \text{ and}$$

$$\Sigma D \geq k(n-2k+2)/(2k-2).$$

Proof. For ease of calculation we will consider the case $n = 2m$. However, the argument for the case n odd is the same, but the calculations are slightly more involved. We give an indirect argument and suppose that at least one of the inequalities in the conclusion of the theorem fails to hold. Let $A = \{v_1, v_2, \ldots, v_m\}$, $B = \{v_{m+1}, v_{m+2}, \ldots, v_{2m}\}$, and $t = d(v_{m+1})$.

Since D has at most k indices, $t \geq m-k+2$. Thus, the fact that the conclusion of the theorem is not true implies that each of the degrees $d(v_{m+i})$ for $1 \leq i \leq m$ are distinct for n sufficiently large. Hence, $d(v_{m+i}) \geq t+i-1$ and v_{m+i} is adjacent to at least $t+i-m$ vertices of A for each $i \geq 1$. Therefore, the number of edges from B to A is at least $1+2+\ldots+t$. On the other hand, the number of edges from A to B is bounded above by the sum of the degrees of the vertices in A. For each $j \in D'$, let r_j be the number of times the degree d_j is duplicated. Thus, $\sum_{j \in D'} r_j = k$. With this notation, we have the following inequality:

$$\sum_{i=1}^{m} d(v_i) \leq 1+2+\ldots+(t-1) + \sum_{j \in D'} (r_j - 1)d_j.$$

Comparing these estimates on the number of edges between A and B gives

$$1+2+\ldots+(t-1) + \sum_{j \in D'} (r_j - 1)d_j \geq 1+2+\ldots+t.$$

If $r = \max\{r_j : j \in D'\}$, then (since $r \leq k$),

$$(r-1)\Sigma D' \geq \sum_{j \in D'} (r_j - 1)d_j, \text{ and}$$

$$\Sigma D' \geq t/(r-1) \geq (n/2 - k + 2)/(r-1) > (n-2k+2)/(2k-2).$$

Also, for n sufficiently large

$$\Sigma D = \sum_{j \in D'} r_j d_j \geq t + \Sigma D' \geq k(n-2k+2)/(2k-2).$$

The last two inequalities give contradictions which complete the proof of Theorem 5. ∎

The bounds on ΣD and $\Sigma D'$ given in Theorem 5 are close to the best possible. This is established by the graph previously denoted as $H(1, (n-k+2)/(2k-2), n-k+1, k)$.

Theorem 6. *Let $k \geq 1$ be a fixed integer, and G be a graph of sufficiently large order n and minimal degree k. Then,*

$$\Sigma D' \geq k, \text{ and}$$

$$\Sigma D \geq k(n+k^2)/(2k+1).$$

Proof. The first inequality is trivial, since any graph must have duplicated degrees and k is the minimal degree. One cannot improve on this, since $\Sigma D' = k$ for a k-regular graph.

Let G be a graph which does not satisfy the second inequality. Since G has minimal degree k, the number, say t, of duplicated degrees is less than $(n+k^2)/(2k+1)$. Select the smallest integer r such that $d_r \geq n-r+1$. Note that such an r exists; in particular, $d_n \geq 1$. For any j, $d_j \geq j+k-t$, since the number of duplicated degrees is t. Thus, if $d_j \leq n-j$, it follows that $j \leq (n+t-k)/2$. This implies $r \leq (n+t-k+2)/2$.

Let $A = \{v_1, v_2, \ldots, v_{r-1}\}$, and let B the remaining vertices of G. For n sufficiently large the degrees of the vertices in B are distinct, since the sum of just two duplicated degrees as large as $d(v_r)$ would exceed $k(n+k^2)/(2k+1)$. Hence, if $d(v_r) = p \geq n-r+1$, then $d(v_{r+i}) \geq p+i-1$ and v_{r+i} is adjacent to at least $p+i-n+r \geq i+1$ vertices of A for each $i \geq 1$. Thus, the number of edges from B to A is at least $1+2+\ldots+(n-r+1)$. On the other hand, the number of edges from A to B is bounded above by the sum of the degrees of the vertices in A, and the only possible degrees are $k, k+1, \ldots, (n-r)$. Therefore, we have the following inequality:

$$\Sigma D \geq (n-r+1)+1+2+\ldots+(k-1).$$

A direct consequence of this and the bounds on r and t imply that $\Sigma D \geq k(n+k^2)/(2k+1)$, a contradiction which completes the proof of Theorem 6. ■

We next describe the second family of graphs that can be derived from the half graph H_n. One of these graphs confirms that the second inequality in Theorem 6 cannot be substantially improved. Other members of this family of graphs will play the same role for the inequalities in Theorem 7 which follows.

Let k be a fixed even positive integer, and let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of the half graph H_n. Assume n is divisible by $2(k-1)$, and let $m = n/2$ and $t = (m/(k-1)) + (k-2)/2$. Alter the graph H_n by deleting the $m-t$ edges between v_m and the vertices $S = \{v_{m+1}, v_{m+2}, \ldots, v_{2m-t}\}$, and adding $m-t$ edges between S and $T = \{v_1, v_2, \ldots, v_{k-2}\}$. This can be done such that each of the vertices of T has degree t and the degrees of the vertices of S are unchanged, since

$$m-t = (k-2)t - \binom{k-1}{2} = (t-1)+(t-2)+\ldots+(t-k+2).$$

For $t \geq k-1$ (which is certainly true for n large) this graph, which we will denote by

$$H_n(k-1,\ (n+(k-1)(k-2))/(2k-2)),\ n-1,\ k),$$

is in $\mathcal{H}_n(k-1,\ (n+(k-1)(k-2))/(2k-2)),\ n-1,\ k)$. Direct calculation gives the following for this graph:

$$\begin{aligned}\Sigma\mathcal{M} &= \binom{k-1}{2},\\ \Sigma D &= k(n+(k-1)(k-2))/(2k-2), \text{ and}\\ \Sigma D' &= (n+(k-1)(k-2))/(2k-2) = t.\end{aligned}$$

Note that the graph $H_n(k,\ (n+k(k-1))/2k),\ n-1,\ k+1)$ has minimum degree k and $\Sigma D = (k+1)(n+k(k-1))/2k$. This indicates that the second inequality in Theorem 6 is the correct order of magnitude.

For the graph $H_n(k-1,\ (n+(k-1)(k-2))/(2k-2)),\ n-1,\ k)$, the minimums of the functions $\Sigma D + \Sigma\mathcal{M}$ and $\Sigma D' + \Sigma\mathcal{M}$, considered as functions of k, can be easily calculated using elementary techniques. Their minimums and the value of k that gives the minimum are the following (for the last function the values are approximated):

$$\begin{array}{lll}\Sigma D'(k) + \Sigma\mathcal{M}(k): & \dfrac{(3(n/2)^{2/3}-1)}{2} & k = (n/2)^{1/3}+1\\ \Sigma D(k) + \Sigma\mathcal{M}(k): & \dfrac{(n+3(n/2)^{2/3})}{2} & k = (n/2)^{1/3}+1\end{array}$$

These give upper bounds for the extremal numbers for each of the functions, and the following theorem will show that they each give the correct value in order of magnitude.

Theorem 7. *If G is a graph of sufficiently large order n, then:*

$$\Sigma D' + \Sigma\mathcal{M} \geq \frac{n^{2/3}}{2}, \text{ and}$$

$$\Sigma D + \Sigma\mathcal{M} \geq n/2 + \frac{n^{2/3}}{2}.$$

Proof. For ease of calculation we will consider the case when $n = 2m$ is even. The argument for n odd is same, although the arithmetic is more complicated. Let $\{v_1, v_2, \ldots, v_m\}$ be the vertices of G which give the degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n$, and let $A = \{v_1, v_2, \ldots, v_m\}$ and B be the remaining vertices of G. Assume that at least one of the inequalities fails to be satisfied by the degree sequence of G.

Clearly any duplicated degree from $m, m+1, \ldots, n-1$ gives $\Sigma D' \geq n/2$ and $\Sigma D \geq n$. Also, any missing degree from $m, m+1, \ldots, n-1$ along with the sum of duplicated degrees given by Theorem 6 implies each of the inequalities of the Theorem. Thus, we assume that none of the degrees $m, m+1, \ldots, n-1$ can be missing or duplicated, so the degrees of the vertices in B are precisely these numbers. This implies for each $i \geq 1$ that $d(v_{m+1}) = m+i-1$ and v_{m+i} is adjacent to at least i vertices of A. Therefore, the number of edges from B to A is at least $1+2+\ldots+m$. On the other hand, the number of edges from A to B is bounded above by the sum of the degrees of the vertices in A. Hence, we have the following inequality (where $S' = S \cap A$):

$$\Sigma S' + \Sigma D \geq 1+2+\ldots+m.$$

No vertex in A has degree $m = n/2$, and therefore

$$\Sigma D \geq n/2 + \Sigma D' + \Sigma \mathcal{M}. \tag{1}$$

If there is a duplicated degree as large as $\frac{n^{2/3}}{2}$, an immediate contradiction is reached. Therefore, each duplicated degrees is less than $\frac{n^{2/3}}{2}$, and so there are more than $n^{1/3}$ indices in D. Since the number of terms in $\mathcal{M}$ or D' is the same as the number of indices in D, we have

$$\Sigma D' + \Sigma \mathcal{M} \geq 1+2+\ldots+\lceil n^{1/3} \rceil > \frac{n^{2/3}}{2}. \tag{2}$$

Inequalities (1) and (2) imply

$$\Sigma D > n/2 + \frac{n^{2/3}}{2}. \tag{3}$$

The inequalities (2) and (3) give contradiction which complete the proof of Theorem 7. ■

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