The Ascending Subgraph Decomposition Problem

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ABSTRACT

Let G be a graph of positive size q, and let n be that positive integer for which $\binom{n+1}{2} \le q < \binom{n+2}{2}$. Then G is said to have an ascending subgraph decomposition if G can be decomposed into n subgraphs $G_1, G_2, \cdots, G_n$ without isolated vertices such that G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \le i \le n-1$. Several classes of graphs possessing an ascending subgraph decomposition are described.

1. Introduction

For graphs F and H, we write $F \subset H$ to indicate that F is isomorphic to a subgraph of H. For definitions and notation not presented here, we follow [1].

It is not difficult to see that every graph G of positive size can be decomposed into subgraphs $G_1, G_2, \cdots, G_k$ without isolated vertices such that $G_1 \subset G_2 \subset \cdots \subset G_k$. (1)

For example, we could let $k = 1$ and choose G_1 to be the graph G less any isolated vertices, or if G has at least two edges, we could let $k = 2$ and define G_1 to be the subgraph induced by an edge e of G and G_2 as $G - e$, less any isolated vertices. A common problem in graph theory is the determination of those graphs G possessing a decomposition (1) such that $G_i \cong H$ $(1 \le i \le k)$ for a given graph H without isolated vertices. (This is referred to as an isomorphic decomposition of G.) In this article, we introduce a problem which is, in a certain sense, opposite to the isomorphic decomposition problem.

Let G be a graph of positive size q. Then there is a maximum number k of subgraphs $G_1, G_2, \cdots, G_k$, without isolated vertices, satisfying (1) such that every two of these subgraphs are nonisomorphic. For such a decomposition then, $|E(G_i)| < |E(G_{i+1})|$ for $1 \le i \le k-1$. Let n be that positive integer for which $\binom{n+1}{2} \le q < \binom{n+2}{2}$. Then n is the maximum number of subgraphs possible in such a decomposition. This motivates the following definition.

Let G be a graph of positive size q, and let n be that positive integer with $\binom{n+1}{2} \le q < \binom{n+2}{2}$. Then G is said to have an <u>ascending subgraph decomposition</u> if G can be decomposed into n subgraphs $G_1, G_2, \cdots, G_n$ without isolated vertices such that G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \le i \le n-1$. A graph G of size $10 = \binom{4+1}{2}$ having an ascending subgraph decomposition G_1, G_2, G_3, G_4 is shown in Figure 1.

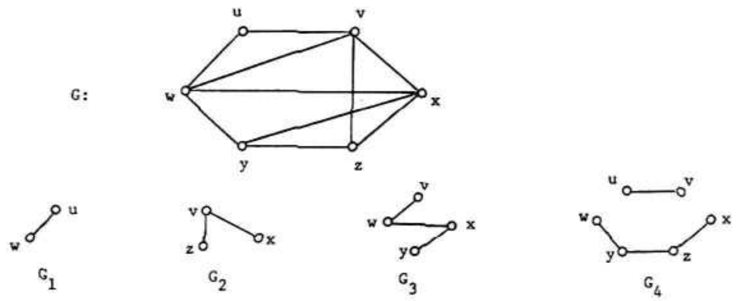

Figure 1 A graph possessing an ascending subgraph decomposition

It is the following problem that is our primary interest.

<u>The Ascending Subgraph Decomposition Problem</u>: Determine those graphs possessing an ascending subgraph decomposition.

If a graph G of size $\binom{n+1}{2}$, for some positive integer n, has an ascending subgraph decomposition $G_1, G_2, \cdots, G_n$, then necessarily G_i has size i for all i $(1 \le i \le n)$. If G has size q, where $\binom{n+1}{2} < q < \binom{n+2}{2}$ for some positive integer n, and has an ascending subgraph decomposition, then G always has such a decomposition where the ith subgraph has size i for $1 \le i \le n-1$, as we now show.

<u>Theorem 1</u> Let G be a graph of size q, where $\binom{n+1}{2} \le q < \binom{n+2}{2}$ for some positive integer n, such that G has an ascending subgraph decomposition. Then G has an ascending subgraph decomposition $G_1, G_2, \cdots, G_n$ such that G_i has size i for $1 \le i \le n-1$ and G_n has size $q - \binom{n}{2}$.

<u>Proof</u> We have already noted that this result is true if $q = \binom{n+1}{2}$; thus, suppose that $\binom{n+1}{2} < q < \binom{n+2}{2}$. By hypothesis, G contains an ascending subgraph decomposition $H_1, H_2, \cdots, H_n$. If H_{n-1} has size $n-1$, then this decomposition has the desired properties. Assume, therefore, that the size of H_{n-1} exceeds $n-1$.

Necessarily, the size of H_1 is 1 or 2. Let e_1 be an edge of H_1, and define G_1 to be the graph induced by e_1. Define G_2 to be the graph induced by any two edges e_2 and e_2' of H_2, so that $G_1 \subset G_2$. Since $H_2 \subset H_3$, the graph H_2 is isomorphic to a subgraph H_3' of H_3. Let $e_3 \in E(H_3) - E(H_3')$ and define $G_3 = \langle E(H_3') \cup \{e_3\} \rangle$, so that G_3 has size 3 and $G_2 \subset G_3$.

Proceeding inductively, we assume that the graphs $G_1, G_2, \cdots, G_k$ have been defined, where $3 \le k < n - 1$, G_k is a subgraph of H_k having size k, and $G_{k-1} \subset G_k$. Since $H_k \subset H_{k+1}$, the graph G_k is isomorphic to a subgraph H_{k+1}' of of H_{k+1}. Let $e_{k+1} \in E(H_{k+1}) - E(H_{k+1}')$ and define $G_{k+1} = \langle E(H_{k+1}') \cup \{e_{k+1}\} \rangle$; thus G_{k+1} has size $k + 1$ and $G_k \subset G_{k+1}$. Therefore, there exist graphs $G_1, G_2, \cdots, G_{n-1}$ such that G_i is a subgraph of H_i having size i for $1 \le i \le n - 1$ and $G_i \subset G_{i+1}$ for $1 \le i \le n - 2$. The proof is completed by defining

$$G_n = \langle E(G) - \bigcup_{i=1}^{n-1} E(G_i) \rangle. \qquad \square$$

We state the following conjecture.

<u>Conjecture</u>. *Every graph of positive size has an ascending subgraph decomposition.*

It suffices to verify this conjecture only for graphs of size $\binom{n+1}{2}$ for $n = 1, 2, \cdots$; for suppose that the conjecture holds for these graphs and that G is a graph of size q, where $\binom{m+1}{2} < q < \binom{m+2}{2}$ for a positive integer m. Let H be a subgraph of G obtained by deleting a set E' of $q - \binom{m+1}{2}$ edges of G. Then H has size $\binom{m+1}{2}$ and, consequently, an ascending subgraph decomposition $H_1, H_2, \cdots, H_m$, where H_i has i edges, $1 \le i \le m$. If we define $G_i = H_i$ for $i = 1, 2, \cdots, m - 1$ and define G_m as that subgraph of G induced by the edge set $E(H_m) \cup E'$, then $G_1, G_2, \cdots, G_m$ is an ascending subgraph decomposition of G.

We now consider some special classes of graphs. For a path or cycle of length $\binom{n+1}{2}$, where $n \ge 2$, the conjecture holds since these graphs can clearly be decomposed into n subgraphs $G_1, G_2, \cdots, G_n$, where G_i is a path of length i $(1 \le i \le n)$. For a complete graph K_{n+1} of size $\binom{n+1}{2}$, there is a natural star decomposition; namely, let G_n denote the star of size n at a vertex of K_{n+1}. If we remove this star, a complete graph K_n results. We then proceed inductively to produce the desired decomposition. It is not difficult to verify that every graph of size 6 has an ascending subgraph decomposition; so we need only consider graphs of size $\binom{n+1}{2}$, where $n \geq 4$.

2. Ascending Subgraph Decompositions into Matchings

We begin this section by showing that every graph with maximum degree at most 2 and size $\binom{n+1}{2}$ has an ascending subgraph decomposition. We may then assume that each such graph is the union of nontrivial paths and cycles. If $n = 2$ and $G \cong C_3$, then $G_1 \cong P_2$, $G_2 \cong P_3$ is an ascending subgraph decomposition of G; while if $n = 3$ and $G \cong 2\,C_3$, then $G_1 \cong P_2$, $G_2 \cong P_3$, $G_3 \cong C_3$ is an ascending subgraph decomposition of G. For every other graph G with maximum degree at most 2 and size $\binom{n+1}{2}$, where $1 \leq n \leq 4$, it is not difficult to verify that G can be decomposed into subgraphs $\{G_i\}$, $1 \leq i \leq n$, such that $G_i \cong i\,K_2$, which is an ascending subgraph decomposition of G. This is also the situation for all such graphs with $n > 4$, as we now show.

Theorem 2 If G is a graph of size $\binom{n+1}{2}$, $n \geq 4$, having maximum degree at most 2, then G has an ascending subgraph decomposition $\{G_i\}$, $1 \leq i \leq n$, such that $G_i \cong iK_2$.

Proof Suppose that the result is false. Then there exists a graph G of minimum size $\binom{n+1}{2}$ having maximum degree 2 and no ascending subgraph decomposition $\{G_i\}$, $1 \leq i \leq n$, such that $G_i \cong iK_2$. From the remark preceding the theorem, $n \geq 5$. Suppose that G is the union of the graphs $F_1, F_2, \cdots, F_k$, where each F_i $(1 \leq i \leq k)$ is a nontrivial path or a cycle. We consider two cases.

Case 1 Suppose that $k \geq n$.

Choose exactly one edge from each of the graphs $F_1, F_2, \cdots, F_n$, and let E_n denote the set of these n (independent) edges. Define $G_n = \langle E_n \rangle$. Then the graph $G - E_n$ has maximum degree at most 2 and size $\binom{n}{2}$. Consequently, $G - E_n$ can be decomposed into subgraphs $G_1, G_2, \cdots, G_{n-1}$ such that $G_i \cong iK_2$ $(1 \leq i \leq n - 1)$. Therefore, $G_1, G_2, \cdots, G_n$ is an ascending subgraph decomposition of G such that $G_i \cong iK_2$ for $1 \leq i \leq n$, contrary to assumption.

Case 2 Suppose that $k < n$.

If, as in Case 1, G has n independent edges, a contradiction is produced. Assume, then, that G does not contain n independent edges. Let a_i $(1 \le i \le k)$ be the maximum number of independent edges of F_i. Then the size of F_i is one of the numbers $2a_i - 1$, $2a_i$, or $2a_i + 1$; in other words, $|E(F_i)| \le 2a_i + 1$. Since G does not contain n independent edges, $\sum_{i=1}^{k} a_i < n$. Further,

$$\binom{n+1}{2} = |E(G)| = \sum_{i=1}^{k} |E(F_i)| \le \sum_{i=1}^{k} (2a_i + 1) = 2\sum_{i=1}^{k} a_i + k < 3n,$$

so that $n < 5$, again producing a contradiction. □

As special cases of the above theorem, we state the following corollary. A <u>linear</u> <u>forest</u> is a forest every component of which is a path.

<u>Corollary</u>. If G is either a linear forest size $\binom{n+1}{2}$, $n \ge 1$, or a union of cycles of size $\binom{n+1}{2}$, $n \ge 4$, then G has an ascending subgraph decomposition $G_1, G_2, \cdots, G_n$ for which $G_i \cong iK_2$ for $1 \le i \le n$.

Of course, we conjecture that every forest of positive size has an ascending subgraph decomposition. In the preceding theorem (and corollary) we have described classes of graphs possessing such decompositions where each subgraph is a matching (i.e., consists of independent edges). We now consider forests having an ascending subgraph decomposition where each subgraph is a matching. For the purpose of doing this, we present three preliminary lemmas.

The maximum degree of a graph G is denoted by $\Delta(G)$ and the edge chromatic number (or chromatic index) of G is denoted by $\chi_1(G)$. We recall the well-known theorem of Vizing [2] that for every graph G with $\Delta(G) \ge 1$, either $\chi_1(G) = \Delta(G)$ or $\chi_1(G) = \Delta(G) + 1$.

<u>Lemma 1</u> If F is a forest with $\Delta(F) \ge 1$, the $\chi_1(F) = \Delta(F)$.

<u>Proof</u> Let $\Delta(F) = d$. Since F is a bipartite graph, F is isomorphic to a subgraph of a d-regular bipartite graph G (see [1, p.284]). Since every regular bipartite graph is 1-factorable (see [1, p.235]), $\chi_1(G) = d$ so that $\chi_1(F) = d$. □

<u>Corollary</u> If F is a forest of size $\binom{n+1}{2}$, where $n \ge 1$, and $\Delta(F) \le (n + 1)/2$, then F contains n independent edges.

<u>Lemma 2</u> If F is a forest with $\Delta(F) = d \ge 2$ and size $\binom{n+1}{2}$, where $n = 2d - 2$, then F contains at most n vertices of degree d.

<u>Proof</u> Suppose that F contains m vertices of degree d. Then F contains at least $m(d - 2) + 2$ end-vertices (see [1, p.75], for example). Hence

$$md + m(d-2) + 2 \le \sum_{v\in V(F)} \deg_F v = 2\binom{n+1}{2} = 2(d-1)(2d-1),$$

implying that

$$2m(d-1) + 2 \le 2(d-1)(2d-1)$$

so that $m(d-1) + 1 \le (d-1)(2d-1)$ or $m(d-1) < (d-1)(2d-1)$. Therefore, $m < 2d - 1 = n + 1$. □

<u>Lemma 3</u> If F is a forest with $\Delta(F) = d \ge 2$ and size $\binom{n+1}{2}$, where $n = 2d - 2$, then F contains a set E of n independent edges such that $\Delta(F - E) = d - 1$.

<u>Proof</u> By Lemma 1, $\chi_1(F) = d$. Let $S_1, S_2, \cdots, S_d$ be the edge color classes in a d-edge-coloring of F. There exists a set S_i $(1 \le i \le d)$ such that

$$|S_i| \ge \left\lceil \binom{n+1}{2}/d \right\rceil = 2d - 2 = n.$$

For each vertex v of degree d, the set S_i contains exactly one edge that is incident with v. By Lemma 2, the forest F contains at most n vertices of degree d. Thus, S_i contains a collection E of n independent edges such that $\Delta(F - E) = d - 1$. □

We are now prepared to present the desired result.

<u>Theorem 3</u> Let n and d be integers with $n \ge 2d - 2 \ge 2$. If F is a forest with $\Delta(F) = d$ and size $\binom{n+1}{2}$, then F has ascending subgraph decomposition $G_1, G_2, \cdots, G_n$ such that $G_i \cong iK_2$ for $1 \le i \le n$.

<u>Proof</u> We proceed by double induction on n and d, where $n \ge 2d - 2 \ge 2$. Let $S(n, d)$ denote the statement: every linear forest F with $\Delta(F) = d$ and size $\binom{n+1}{2}$ has an ascending subgraph decomposition $G_1, G_2, \cdots, G_n$ such that $G_i \cong iK_2$ for $1 \le i \le n$. By Theorem 2, $S(n, 2)$ is true.

For $d > 2$, assume that $S(m, d-1)$ is true for all $m \ge 2(d-1) - 2$. We show that $S(n, d)$ is true for all $n \ge 2d - 2$.

First we verify that $S(2d - 2, d)$ is true. By Lemma 3, it follows that if F is a forest with $\Delta(F) = d$ and size $\binom{n+1}{2}$, where $n = 2d - 2$, then F contains a set E_n of n independent edges such that $F' = F - E_n$ has maximum degree $d - 1$. Thus, F' has size $\binom{n}{2}$ and $\Delta(F') = d - 1$. Since $n - 1 = 2d - 3 > 2(d - 1) - 2$ and $S(n - 1, d - 1)$ is true, F' has an ascending subgraph decomposition $G_1, G_2, \cdots, G_{n-1}$ such that $G_i \cong i K_2$ for $1 \le i \le n - 1$. Letting $G_n = \langle E_n \rangle$, we see that $S(2d - 2, d)$ is true.

Let n be an integer such that $n > 2d - 2$ and assume that $S(n - 1, d)$ is true. We prove that $S(n, d)$ is true. Let F be a forest with $\Delta(F) = d$ and size $\binom{n+1}{2}$. Since $d \le (n + 1)/2$, it follows from the corollary to Lemma 1 that F contains a set E_n of n independent edges. The forest $F' = F - E_n$ has size $\binom{n}{2}$. If $\Delta(F') = d$, then since $n - 1 \ge 2d - 2$ and $S(n - 1, d)$ is true, it follows that F' has an ascending subgraph decomposition, $G_1, G_2, \cdots, G_{n-1}$ such that $G_i \cong i K_2$ for $1 \le i \le n - 1$. That $S(n, d)$ is true now follows by letting $G_n = \langle E_n \rangle$. Suppose now that $\Delta(F') = d - 1$. Since $n - 1 \ge 2d - 2 > 2(d - 1) - 2$ and $S(n - 1, d - 1)$ is true, F' has an ascending subgraph decomposition $G_1, G_2, \cdots, G_{n-1}$ such that $G_i \cong i K_2$ for $1 \le i \le n - 1$. The proof that $S(n, d)$ is true is completed by defining $G_n = \langle E_n \rangle$.

□

We now show that the bound $n \ge 2d - 2$ presented in the previous theorem cannot be improved in general. For $d \ge 3$, let $n = 2d - 3$. Then $\binom{n+1}{2} = (d - 1)(2d - 3)$. Let G be the forest that consists of $n - 1$ copies of the star $K(1, d)$. The size of G is $(n - 1)d$. Since $d \ge 3$, it follows that $(n - 1)d \ge \binom{n+1}{2}$. Delete $(n - 1)d - \binom{n+1}{2}$ edges from G to produce a forest F of size $\binom{n+1}{2}$. Since G does not contain n independent edges, neither does F. Hence F does not contain an ascending subgraph decomposition $G_1, G_2, \cdots, G_n$ such that $G_i \cong i K_2$ for $1 \le i \le n$. Of course, this does not present a counterexample to our conjecture. This only says that any ascending subgraph decomposition of F is of a different type.

3. Some Concluding Remarks

We have conjectured that every graph G of size $\binom{n+1}{2}$, $n \geq 1$, can be decomposed into subgraphs $G_1, G_2, \cdots, G_n$ such that G_i has size i $(1 \leq i \leq n)$ and $G_1 \subset G_2 \subset \cdots \subset G_n$. Whether this conjecture is true or not, there is a class of related problems produced by adding restrictions to the subgraph G_i. For example, for $n \geq 1$, determine functions $f(n)$ and $g(n)$ such that if $\Delta(G) \leq f(n)$, then G has an ascending subgraph decomposition $G_1, G_2, \cdots, G_n$ with $\Delta(G_n) \leq g(n)$. By Theorem 3, if G is a forest, then $f(n) = (n + 2)/2$ and $g(n) = 1$.

Another problem is to determine which graphs G of size $\binom{n+1}{2}$, $n \geq 1$, possess an ascending connected subgraph decomposition, i.e., each G_i is connected. Special cases of this would be to require all subgraphs G_i to be trees or, even more restrictively, to require all subgraphs G_i to be stars. In connection with this last problem, we present a number-theoretic conjecture.

<u>Conjecture</u> Let $n \geq 2$ be an integer, and let $a_1, a_2, \cdots, a_k$ be integers such that $n \leq a_i \leq 2n - 2$ and $\sum_{i=1}^{k} a_i = \binom{n+1}{2}$. Then there exists a partition of the set $S = \{1, 2, \cdots, n\}$ into k subsets $S_1, S_2, \cdots, S_k$ such that for each i $(1 \leq i \leq k)$ $a_i = \sum_{j \in S_i} j$.

This conjecture is equivalent to the following graph theoretic conjecture.

<u>Conjecture</u>. Let $n \geq 2$ be an integer and let G be a union of stars $S_1, S_2, \cdots, S_k$, such that S_i has size a_i, where $n \leq a_i \leq 2n - 2$, and G has size $\sum_{i=1}^{k} a_i = \binom{n+1}{2}$. Then G has an ascending star decomposition.

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