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SOME REMARKS ON INFINITE SERIES

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Dedicated to Professor K. Tandori on the occasion of his 60th birthday

In the present paper we investigate the following problems. Suppose $a_n > 0$ for $n \geqq 1$ and $\sum_{n=1}^{\infty} a_n = \infty$.

N° 1. Does there exist a sequence of natural numbers $N_0 = 0$, $N_i \nearrow \infty$, such that it decomposes the series monotone decreasingly:

$$\sum_{j=N_i+1}^{N_{i+1}} a_j \geqq \sum_{j=N_{i+1}+1}^{N_{i+2}} a_j \qquad (i = 0, 1, 2, \ldots)? \tag{1}$$

In order to state the second problem we define the index $n_k(c)$ as the minimum m such that

$$kc \leqq \sum_{j=1}^{m} a_j. \tag{2}$$

Now the second problem is as follows.

N° 2. What is the relation between the behaviour of $\sum_{1}^{\infty} a_n^2$ and the typical behaviour of $\sum_{k=1}^{\infty} a_{n_k(c)}$ (c is variable)? As it turns out, the two problems are related. Problem N° 1 is motivated by the fact, that for every non-negative continuous function $f\colon [0, \infty) \to \mathbf{R}$ it is easy to define a sequence $x_i \nearrow \infty$ such that $\int_{x_n}^{x_{n+1}} f \geqq \int_{x_{n+1}}^{x_{n+2}} f$ $(n = 0, 1, \ldots)$.

THEOREM 1. *Suppose $a_n > 0$, $a_n \geqq a_{n+1}$ for every $n \geqq 1$, $\sum_{n=1}^{\infty} a_n = \infty$. Then for every $c > 0$*

$$\sum_{n=1}^{\infty} a_n^2 \quad \textit{and} \quad \sum_{k=1}^{\infty} a_{n_k(c)}$$

are equiconvergent.

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PROOF.[1] We may suppose $a_n \searrow 0$, since in the opposite case the statement is trivial. Hence we have for $k > K(c)$

$$n_{k+1}(c) > n_k(c)$$

and

$$\sum_{i=n_k(c)+1}^{n_{k+1}(c)} a_i = c + o(1).$$

In view of monotonicity of (a_n) for $k > K(c)$

$$\Big(\sum_{i=n_k(c)+1}^{n_{k+1}(c)} a_i\Big) a_{n_k(c)} \geqq \sum_{i=n_k(c)+1}^{n_{k+1}(c)} a_i^2 \geqq \Big(\sum_{i=n_k(c)+1}^{n_{k+1}(c)} a_i\Big) a_{n_{k+1}(c)},$$

and the equiconvergence holds. ∎

Theorem 1 makes possible to give a partial solution for problem N° 1.

THEOREM 2. *Suppose* $a_n > 0$, $\sum_{n=1}^{\infty} a_n = \infty$.

(i) *If* (a_n) *has a majorant* $(b_n) \in l_2$ *with* $b_n \geqq b_{n+1}$ *for* $n \geqq 1$, *then* $\sum a_n$ *has the decomposition required in* (1).

(ii) *If* $a_n \geqq a_{n+1}$ *for* $n \geqq 1$, $(a_n) \notin l_2$, *then there exists a series* $\sum b_n$ *having no decomposition and* $1/3 < a_n/b_n < 3$.

PROOF. In the first step we prove the existence of the required decomposition (1) for (b_n). Let $N_0 = 0$. We define N_1 so large, that

$$K_1 := \sum_{j=1}^{N_1} b_j$$

obeys

$$K_1/6 > \max_n b_n \tag{3}$$

$$\sum_{k=1}^{\infty} b_{n_k(K_1/3)} < K_1/2. \tag{4}$$

The number N_1 exists, since $\sum_{k=1}^{\infty} b_{n_k(c)}$ is finite by Theorem 1 and monotone decreasing in c, and K_1 is as large as we want.

Suppose $N_0, N_1, \dots, N_i, N_{i+1}$ are defined and

$$K_i := \sum_{j=N_i+1}^{N_{i+1}} b_j \geqq K_1/2.$$

Let N_{i+2} be the largest index for which

$$\sum_{j=N_i+1}^{N_{i+1}} b_j \geqq \sum_{j=N_{i+1}+1}^{N_{i+2}} b_j.$$

[1] The present simple proof is due to G. Petruska.

By (3) we have $N_{i+2}>N_{i+1}$. We prove $K_{i+1}:=\sum_{j=N_{i+1}+1}^{N_{i+2}} b_j \geqq K_1/2$, what means, N_i and K_i are defined for $i>0$ with $K_1 \geqq K_2 \geqq K_3 \geqq \dots$.

Assume m is the least integer with $K_{m+1}<K_1/2$. First, $K_m \geqq K_1/2$ and by the choice of N_i's and by (3) $K_m-K_{m+1}<K_1/6$, hence $K_{m+1} \geqq K_1/3$. On the other hand

$$K_1-K_{m+1} \leqq \sum_{i=0}^{m+1} b_{N_{i+1}+1} \leqq \sum_{k=1}^{m+1} b_{n_k(K_{m+1})} \leqq \sum_{k=1}^{\infty} b_{n_k(K_1/3)}.$$

Using (4) we have $K_{m+1} \geqq K_1/2$, a contradiction.

In the second step set $M_0=0$, select M_1 so large that

$$K_1 < \sum_{j=1}^{M_1} a_j$$

and let M_{i+2} be the largest integer with

$$\sum_{j=M_i+1}^{M_{i+1}} a_j \geqq \sum_{j=M_{i+1}+1}^{M_{i+2}} a_j.$$

Set

$$L_i:=\sum_{j=M_i+1}^{M_{i+1}} a_j.$$

We have to prove $M_{m+2}>M_{m+1}$ for $m>0$. Obviously, $M_i \geqq N_i$ and

$$L_1-L_{m+1} \leqq \sum_{i=0}^{m+1} a_{M_{i+1}+1} \leqq \sum_{i=0}^{m+1} b_{M_{i+1}+1} \leqq \sum_{i=0}^{m+1} b_{N_{i+1}+1} \leqq \sum_{k=1}^{\infty} b_{n_k(K_1/2)} < K_1/2,$$

what means $L_{m+1}>K_1/2$, i.e. $M_{m+2}>M_{m+1}$. In order to prove (ii) suppose without loss of generality $a_1<1$ and set $f(0)=0$,

$$f(n):=\left|\{k:\ 2^{-n} \leqq a_k < 2^{-n+1}\}\right|$$

for $n \geqq 1$. It is well-known that

$$\sum_{n=1}^{\infty} a_n^2 < \infty$$

if and only if $\sum_{n=1}^{\infty} f(n)4^{-n}<\infty$. If $f(n)>0$ we define a strictly monotone increasing sequence $\varepsilon_{n,j}$ $(j=1, 2, \dots, f(n))$ obeying $0 \leqq \varepsilon_{n,j} \leqq 4^{-n}$. For every natural number i there exists a unique m with

$$f(0)+f(1)+\dots+f(m-1) < i \leqq f(0)+f(1)+\dots+f(m).$$

We define

$$b_i:=2^{-m}+\varepsilon_{m,\, i-\sum_{j=0}^{m-1} f(j)}, \tag{5}$$

and prove that $\sum b_i$ satisfies the requirements of (ii). Obviously, $1/3<a_n/b_n<3$. The sequence (b_i) is monotone increasing in the intervals

$$\left(\sum_{j=0}^{m-1} f(j),\ \sum_{j=0}^{m} f(j)\right]$$

of indices, by (5).

Suppose there exists a decomposition required in (1) for $\sum b_i$ with indices $N_0=0<N_1<N_2<\dots$ and

$$K_i = \sum_{j=N_i+1}^{N_{i+1}} b_j.$$

We are going to prove $K_1=\infty$, a contradiction. If

$$\sum_{j=0}^{m-1} f(j) \leqq N_i < N_{i+1} < \sum_{j=0}^{m} f(j) \tag{6}$$

then $K_i-K_{i+1}\geqq 2^{-m}$, since $N_{i+2}-N_{i+1}<N_{i+1}-N_i$ by the strictly monotone increasingness of (b_i) in the above considered interval. Since $K_1\geqq K_2\geqq K_3\geqq\dots$ by (1), we have

$$\left|\{i\colon (6) \text{ holds for } i\}\right| \geqq \frac{f(m)}{K_1\cdot 2^m}-3.$$

Comparing our estimates we have

$$K_1 \geqq \sum_{i=0}^{\infty}(K_i-K_{i+1}) \geqq \sum_{(6)\text{ holds for } i}(K_i-K_{i+1}) \geqq \sum_m 2^{-m}\left(\frac{f(m)}{K_1\cdot 2^m}-3\right)=\infty. \quad \blacksquare$$

M. Szegedy noted, that with a bit more effort one can prove (ii) with $b_i=a_i(1+o(1))$. We have conjectured that $(a_n)\in l_2$ is sufficient for having a decomposition. Recently, the conjecture was proved by M. Szegedy and G. Tardos [1].

Now we investigate what happens if we drop the condition $a_n\geqq a_{n+1}$ from Theorem 1. It is clear, that dropping the condition a counterexample can be given for a fixed c, but we have

THEOREM 3. *Suppose* $a_n>0$, $\sum_{n=0}^{\infty} a_n=\infty$. *If*

$$\sum_{n=0}^{\infty} a_n^2<\infty, \quad \textit{then} \quad X:=\left\{c\colon \sum_{k=1}^{\infty} a_{n_k(c)}=\infty\right\}$$

is of measure zero, and if

$$\sum_{n=0}^{\infty} a_n^2=\infty, \quad \textit{then} \quad Y:=\left\{c\colon \sum_{k=1}^{\infty} a_{n_k(c)}<\infty\right\}$$

is meagre (i.e. of first category).

PROOF. In the first case we have for $0<a<b<\infty$

$$\sum_{k=1}^{\infty}\int_a^b a_{n_k(c)}\,dc<\infty,$$

what proves the first statement by Beppo Levi's theorem. Indeed, we have for $k>K(c)$

$$\int_a^b a_{n_k(c)}\,dc \leqq \frac{1}{k}\sum_{ka\leqq \sum_{i=1}^{j} a_i<kb} a_j^2$$

and

$$\sum_{k=1}^{\infty}\int_a^b a_{n_k(c)}\,dc \leqq \sum_{j=1}^{\infty} a_j^2 \sum_{\frac{1}{b}\sum_{i=1}^{j} a_i \leqq k < \frac{1}{a}\sum_{i=1}^{j} a_i} \frac{1}{k} = \sum_{j=1}^{\infty} a_j^2\left(\log\frac{b}{a}+o(1)\right) < \infty.$$

In the second case we prove for $0<a<b<\infty$

$$\sum_{k=1}^{\infty}\int_a^b a_{n_k(c)}\,dc = \infty.$$

It is trivial, if $\inf_n a_n=\varepsilon>0$. If not, the previous estimates will be repeated for $a<a'<$ $<b'<b$ in the inverse direction and

$$\sum_{k=1}^{\infty}\int_a^b a_{n_k(c)}\,dc \geqq \sum_{j=1}^{\infty} a_j^2\left(\log\frac{b'}{a'}+o(1)\right) = \infty.$$

The function $c\to f(c):=\sum_{k=1}^{\infty} a_{n_k(c)}$ is lower semicontinuous from the left side since $\lim_{c\to c_0-} f(c)\geqq f(c_0)$, so

$$H_i:=\left\{c:\ \sum_{k=1}^{\infty} a_{n_k(c)} > i\right\}$$

contains a dense open set $G_i\subset(0,\infty)$. This way

$$\left\{c:\ \sum_{k=1}^{\infty} a_{n_k(c)}=\infty\right\} = \bigcap_i H_i \supset \bigcap_i G_i$$

and

$$\left\{c:\ \sum_{k=1}^{\infty} a_{n_k(c)}<\infty\right\}$$

is meagre. ∎

The size of an exceptional set in Theorem 3 is still an open question. A particular answer is given by the next construction.

THEOREM 4. *X can be residual, and Y can be of cardinality continuum.*

PROOF. We construct $\sum_{n=1}^{\infty} a_n^2<\infty$ with a residual X. Suppose $\{\alpha_i:\ i\in\mathbf{N}\}$ is dense in $(0,\infty)$ and let β_i be $\beta_i=\alpha_{i-\binom{k}{2}}$ if $\binom{k}{2}<i\leqq\binom{k+1}{2}$. For every β_i set some segments $a_j:\ j\in I_i$, so, that

— I_i finite, $a_j:\ j\in I_i$ are disjoint,

— on the ray $(0,\infty)$ all $a_j:\ j\in I_i$ is on the right hand from all $a_j:\ j\in I_k$, where $k<i$,

— $\sum_{j\in I_i} a_j^2<\frac{1}{2^i}$, $\sum_{j\in I_i} a_j\geqq 1$,

— all the segments a_j have in their interior a multiple of β_i.

We cover the rest of the ray with segments $a_j:\ j\in J$ such that $\sum_{j\in J} a_j^2<\infty$.

If β_i is the n-th repetition of α_k, there is a neighbourhood V_k^n of α_k, such that $m_j\alpha_k\in a_j$ $(m_j\in\mathbf{N})$ implies $m_jV_k^n\subset a_j$ $(j\in I_i)$. Now clearly $\bigcap_n(\bigcup_k V_k^n)$ is residual and X contains it.

Now we construct a perfect set Y (i.e. of cardinality continuum) in the following way. Set $I_0^1=[100, 101]$, we are going to define closed intervals I_n^i $(i=1, ..., 2^n)$ for $n=1, 2, ...$ with the property: I_n^i contains the disjoint intervals I_{n+1}^{2i} and I_{n+1}^{2i-1}. We have a perfect set $\bigcap_n(\bigcup_i I_n^i)=Y$. In $\bigcup_i I_n^i$ we select 2^{n+1} numbers $x_1, ..., x_{2^{n+1}}$ independent over the field of rationals, two of which are in $\operatorname{int} I_n^i$ $(i=1, ..., 2^n)$. By Kronecker's Theorem for infinitely many α_j

$$|\alpha_j-k_{i,j}x_i|<0{,}001$$

for $i=1, 2, ..., 2^{n+1}$, $k_{i,j}$ integer. We are interested only in $\alpha_1, ..., \alpha_n$. We set an interval $J_m^{(n)}$ $(m=1, ..., n)$, $|J_m^{(n)}|=1/200$ close to α_j but right to it, $J_m^{(n)}$ not containing any multiple of $x_1, x_2, ..., x_{2^{n+1}}$, right from the previous $J_i^{(l)}$ $(l<n; 1\le i\le 2^l)$. Now we define I_{n+1}^i as short intervals centered at x_i, so that none of the $J_m^{(n)}$ $(m=1, ..., n)$ intersect any multiple of I_{n+1}^i. Finally we define the series $\sum_{n=1}^{\infty} a_n$. All the intervals $J_m^{(n)}$ $(n=1, 2, ...; m=1, 2, ..., n)$ occur as some $a_{s(n,m)}$ with

$$\sum_{i=1}^{s(n,m)} a_i=\text{the right endpoint of } J_m^{(n)}.$$

The "undefined gaps" in $\sum a_n$ we fill with small numbers tending quickly to zero.

It is easy to check, that $\sum a_n=\infty$, $\sum a_n^2=\infty$, since $a_n\not\to 0$. $c\in Y$ implies $\sum a_{n_k(c)}<\infty$, since the multiples of c avoid all the intervals $J_m^{(n)}$.

REMARK. With a little care we can construct a series with the above properties with $a_n\to 0$.

PROBLEM 1. Is there a topological property φ such that

$$\{c: \sum a_{n_k(c)}<\infty\}\in\varphi \quad \text{if and only if} \quad \sum a_n^2<\infty?$$

PROBLEM 2. Is there a series $\sum a_n^2<\infty$ in Theorem 3 with Y of positive measure?

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