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SOME PROBLEMS ON FINITE AND INFINITE GRAPHS

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Many of the problems Hajnal and I posed 15 years ago have been solved positively or negatively or shown to be undecidable. I state some of the remaining ones and add a few new ones. I do not give a complete list of references but add a few where it seems essential.

1. Determine the α's for which $\omega^\alpha \to (\omega^\alpha, 3)^2$. α must be a power of ω. I offer \$1000 for a complete characterization and \$250 for $\alpha = \omega^2$, the first open case. Jean Larson, A short proof of a partition theorem for the ordinal ω^ω, Ann. Math. Logic 6(1973/74).

2. Is it true that if $\alpha \to (\alpha, 3)_2^2$ then also $\alpha \to (\alpha, n)_2^2$?

3. $c \to (\omega+n, 4)_2^3$ is an old result of Rado and myself. $\omega+n$ can perhaps be replaced by any $\alpha < \omega_1$ and 4 by any $n < \omega$, but I know nothing about this.

Hajnal and I proved $\omega_1^2 \to (\omega_1\omega, 3)^2$. To our annoyance we could never show $\omega_1^2 \to (\omega_1\omega, 4)_2^2$. During this meeting Baumgartner and Hajnal proved, assuming the continuum hypothesis

$$\omega_1^2 \not\to (\omega_1\omega, 4)^2$$

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Their paper about it will appear in this volume. They also proved $\omega_1^2 \to (\omega_1\omega,3,3)_1^2$. Also we could never decide $\omega_2\omega \to (\omega_2\omega,3)_2^2$. (I just heard [1985, October] that S. Shelah just proved this and many other of our conjectures. Perhaps if G is any graph of power $\aleph_1$ which contains no K_4 then $\omega_1^2 \to (\omega_1\omega,G)^2$. This is open even if G is assumed to be finite. (Baumgartner just showed that $\omega_1^2 \not\to (\omega_1\omega,K(\aleph_0,\aleph_0))^2$. Perhaps $\omega_1^2 \to (\omega_1\omega,G)^2$ holds if G contains no $K(4)$ and no $K(\aleph_0,\aleph_0)$, perhaps it will be necessary to restrict G to have finite order.)

4. Is it true that if G_1 and G_2 are $\aleph_1$-chromatic graphs then they have a common 4-chromatic subgraph? Perhaps they have a common $\aleph_0$-chromatic subgraph as well. On the other hand we can not at present exclude the possibility that there are $\aleph_0$ graphs G_n, $n < \omega$ such that G_n has chromatic number $\aleph_1$ and no two of them have a common 4-chromatic subgraph. It seems more likely that any finite set of $\aleph_1$-chromatic graphs have a common 4-chromatic ($\aleph_0$-chromatic?) subgraph. Probably every $\aleph_1$-chromatic graph contains all 4-chromatic subgraphs all circuits of which are of length $> n_G$. An old result of Hajnal, Shelah and myself shows this for three-chromatic graphs.

5. An old problem of Hajnal and myself states: Is there a graph G which contains no K_4 and which is not the union of $\aleph_0$ graphs which are triangle free? I offer 250 dollars for this problem. Folkman Nesetril and Rödl proved that for every n there is a G which contains no K_4 and is not the union of n triangle free graphs. For a while Hajnal and I thought that perhaps the following result would hold: if G_1 and G_2 are two graphs so that for every $n < \omega$ there is a graph G_n which contains no G_1 but if we color the edges of G_n by n colors then at least one of the colors contains G_2, then the same result holds if n is $\aleph_0$ and in fact every infinite cardinal number. This

certainly fails if G_1 is C_4 and G_2 is C_6 (or in fact any bipartite graph not containing C_4). Hajnal and I proved that every graph which contains no C_4 is the denumerable union of trees and Nesetril and Rödl proved that for every n there is a graph not containing C_4 which is not the union of n graphs not containing C_6. It would be perhaps of interest to decide for which G_1 and G_2 does our original guess hold? The most interesting case is of course $G_1 = K_4$, $G_2 = K_3$. The paper of Nesetril and Rödl will soon appear in Trans. Amer. Math. Soc.

6. Is it true that if $f(n)$ increases arbitrarily fast then there is an $\aleph_1$-chromatic G so that if $g(n)$ is the smallest integer for which G has an n-chromatic subgraph of $g(n)$ vertices then $f(n)/g(n) \to 0$?

7. Hajnal, Szemeredi and I have the following problem. Let $h(n) \to \infty$ as slowly as we please. Is it true that there is a G of chromatic number $\aleph_0$ so that any subgraph of n vertices of G can be made two-chromatic by the omission of $h(n)$ edges? P. Erdös, A. Hajnal and E. Szemeredi, On almost bipartite large chromatic graphs, Annals of Discrete Math. Vol. 12, Theory and Practice of Combinatorics, Dedicated to A. Kotzig 114-123.

If G has chromatic number $\aleph_1$ it is easy to see that this is not true with $h(n) = cn$ (c small) and we conjecture that for any such $h(n)$, $h(n)/n \to \infty$. We proved that there is a G for which $h(n) < n^{3/2}$ and in fact by the omission of $n^{1+\varepsilon_k}$ edges any set of n vertices can be made to have chromatic number $\leq k$, $\varepsilon_k \to 0$ as $k \to \infty$. However we have no guess of the true order of magnitude of $h(n)$. V. Rödl, Nearly bipartite graphs with large chromatic number, Combinatorica 2(1982), 377-383.

8. Is it true that for every infinite m one can color the countable subsets of m by $(2^{\aleph_0})^+$ colors so that every subset of size

$(2^{\aleph_0})^+$ gets subsets of all the colors?

9. An old conjecture of mine on graphs states as follows: Let $\mathcal{G}$ be a graph. A and B are two disjoint independent sets of $\mathcal{G}$. S separates A from B if every path joining a vertex of A to a vertex of B passes through a vertex of S. My conjecture now states that S can be chosen so that through every vertex of S there is a path joining A and B so that these paths should be vertex disjoint. If $|S| < \aleph_0$ then this is the well known theorem of Menger. For $|S| = \aleph_0$ the problem is open and could very well be false. Recently important work on this problem has been done by Aharoni who settled the problem if $\mathcal{G}$ is bipartite, but the general problem is still open.

10. Another old problem of Hajnal, Milner and myself stated: Let α be an ordinal number which has no immediate predecessor (i.e. is a limit number). For which α is it true that if $\mathcal{G}$ is a graph whose vertices form a set of type α then either G has an infinite path or contains an independent set of type α. We proved this for $\alpha < \omega_1^{\omega+2}$. Recently important work on this problem was done by Jean Larson and Baumgartner; the papers will soon appear. In particular it is consistent that for every $\alpha < \omega_2$, $\alpha \not\rightarrow (\omega_1^{\omega+2}$, infinite path), but the general problem $\alpha \not\rightarrow (\beta$, infinite path) is still open.

11. Now I want to mention a few recent finite problems: Bruce Rothschild and I recently considered the following problem: Let $\mathcal{G}(n;e)$ be a graph of n vertices and e edges, $e \geq cn^2$. Assume further that every edge of $\mathcal{G}$ is contained in at least one triangle. Define $f(n;c)$ as the smallest integer so that in every such graph there is an edge contained in at least $f(n;c)$ triangles. Estimate $f(n;c)$ as well as possible. Noga Alon showed that $f(n;c) < \alpha_c\sqrt{n}$ and Szemeredi observed that his regularity Lemma implies $f(n;c) \to \infty$ for every $c > 0$. Is it true that $f(n;c) > n^{\varepsilon}$ (or at least $f(n;c) > \log n$)?

More generally the following problem is of interest. Let $e(n,r)$ be the smallest integer so that every $\mathcal{G}(n;e(n,r))$ each edge of which is contained in at least one triangle has an edge which is contained in at least r triangles. Ruzsa and Szemerédi proved

$$cnr_3(n) < e(n;2) = \sigma(n^2)$$

where $r_3(n)$ is the largest integer so that there is a set of $r_3(n)$ integers $< n$ not containing an arithmetic progression of three terms. $e(n;r) = \sigma(n^2)$ holds for every r as stated previously. Probably $e(n;r+1) - e(n;r) \to \infty$. But perhaps $e(n;r+1)/e(n;r) \to 1$. I. Ruzsa and E. Szemerédi, Triple systems with no three points carrying three triangles, Coll. Math. Soc. J. Bolyai Vol. 18 Combinatorics, 939-945.

12. Finally I would like to state a few problems of Péter Komjáth which have the attractive property that they seem to be easy (almost trivial), but are perhaps difficult.

Let $|A_i| = \aleph_0$, $|A_i \cap A_j| < \aleph_0$ and $|A_i \cap A_j| \neq 1$. Is such a family necessarily two-chromatic?

Let A_i be a family of denumerable sets $|A_i \cap A_j| \neq 2$. Is there a bound on the chromatic number of such a family? If instead of $|A_i \cap A_j| \neq 2$, $|A_i \cap A_j| \neq 1$ is assumed Komjáth easily showed that the chromatic number of this family is at most $\aleph_0$.

References

P. Erdös and A. Hajnal, Unsolved problems in set theory, Proc. Symp. Pure Math. XIII part 1 Amer. Math. Soc. 1971, 17-43 and Solved and unsolved problems in set theory, Proc. Tarski Symposium Proc. Symp. Pure Math Vol. XXV (1974), 269-287.

P. Erdös, Problems and results on finite and infinite combinatorial analysis, Coll. Math. Soc. J. Bolyai 1973, 403-424. This is a much less systematic account than our papers with Hajnal. For a very rich source of solved and unsolved problem see P. Erdös, F. Galvin and A. Hajnal, On set systems having large chromatic number and not containing prescribed sub-systems, ibid. 425-513.