REMARKS ON STARS AND INDEPENDENT SETS

P. Erdös and J. Pach
Mathematical Institute of the Hungarian Academy of Sciences

1 INTRODUCTION

Let G be a graph with vertex set and edge set V(G) and E(G), respectively. A subset $X \subseteq V(G)$ is said to be *independent* if there are no two elements of X connected by an edge of G. We say that G has *property I* if for every independent set X there is a common neighbour in G, i.e. we can find a point y_X such that $y_X x \in E(G)$ for every $x \in X$. The neighbourhood $\{v \in V(G) | yv \in E(G)\}$ of a vertex y is usually called a *star* of the graph. In this terminology, property I means that every independent set of G is covered by a star.

A couple of years ago Erdös and Fajtolowicz [2] conjectured that there exists a positive constant ε such that every *triangle-free* graph G on n vertices having property I has a vertex of degree at least εn. Note that such a graph can be characterized by saying that the system of maximal (non-increasable) independent sets of G coincides with the system of stars. The above conjecture has been settled by Pach [6] who proved the following.

Theorem 1. Every triangle-free graph on n vertices having property I contains a vertex degree at least $\frac{n+1}{3}$. This result is best possible, provided $3|n+1$.

Pach completely characterized all the graphs having these properties. He ended his paper with the question whether the same assertion holds if we drop the condition that the graphs are triangle-free. Our Theorem 2 answers this question in the negative. We can construct a graph of property I with maximum degree $(1+o(1))n \log\log n/\log n$, and we shall show that this bound cannot be improved.

What happens now if, instead of the condition that our graphs are triangle-free, we assume only that they do not contain K_r (a complete subgraph on r vertices), for some $r > 3$? In particular, is it true that every K_4-free graph of n vertices having property I contains a point of degree at least εn? Unfortunately, we can throw very little light on this simple question. Our only result in this direction (Theorem 3) is that, if $w(n)$ is any function tending to infinity as slowly as we please, then there exist graphs on n vertices containing no $K_{w(n)\log n}$, having property I, but whose maximum degree is $o(n)$.

In what follows, we need the following

Definition 1. Given any natural number k, a graph G is said to have property I_k if every independent set of cardinality k has a common neighbour in G.

Obviously, a graph has property I if and only if it has property I_k for all k. Property I_2 means that G has diameter 2.

We can sharpen Theorem 1 by proving

Theorem 4. Every triangle-free graph on n vertices and with property $I_{\lceil \log n \rceil}$ has a vertex of degree at least $\frac{n+1}{3}$.

Erdös and Fajtlowicz suspected that this result remains true (apart from the constant factor beside n) even if $I_{\lceil \log n \rceil}$ is replaced by the (essentially weaker) property I_3, but this was disproved in [6]. However, at the moment we have no idea how to attack the next question to arise naturally: Does there exist an $\varepsilon > 0$ such that every triangle-free graph with property I_4, having n vertices, contains a point of degree $\geq \varepsilon n$?

Let $f_{I_k}(r,n)$ denote the maximum integer f such that every K_r-free graph of n vertices having property I_k has a vertex of degree at least f.

It is not difficult to show (cf. [6]) that, for any fixed k and r ($k \geq 2$, $r \geq 3$), we have

$$f_{I_k}(r,n) \geq (1 - o(1))n^{1-(1/k)}.$$

The great weakness of our results is that we cannot improve on this lower bound for any pair (k,r). The simplest case $k=2$, $r=3$ was considered by Erdös and Fajtlowicz [2], who proved

$$\sqrt{n}\log n \geq f_{I_2}(3,n) \geq \sqrt{n-1} .$$

We conjecture that here the *upper* bound is not far from the truth. On the other hand, a well-known construction of Erdös-Rényi [4,5] and Brown [1], using finite projective planes, shows that, in case $k=2$, $r>3$, the *lower* bound is asymptotically sharp. An extension of their construction (see [6]) proves that, in general,

$$f_{I_k}(k+2,n) = (1+o(1))n^{1-(1/k)}$$

holds for every $k \geq 2$.

Next we show that $f_{I_k}(k+1,n) = o(n)$. Let X be a $((k+1)m-1)$ - element set, and define a graph G whose vertices are the m-element subsets of X, two of them being joined by an edge if and only if their intersection is empty. It is now clear that G has property I_k and does not contain a complete subgraph on $k+1$ vertices.

Setting $n = |v(G)| = \binom{(k+1)\cdot m-1}{m}$, easy calculation shows that the degree $\binom{k\cdot m-1}{m}$ is $o(n)$. More exactly,

$$f_{I_k}(k+1,n) \leq (1+o(1))n^{\frac{k\log k-(k-1)\log(k-1)}{(k+1)\log(k+1)-k\log k}}$$

is valid, for every $k \geq 2$.

We almost certainly have that, if k is odd, then $f_{I_k}(k,n) = o(n)$, but we have not yet worked out the details. (The parity condition seems to be merely a technical requirement.)

Definition 2. ([3].) Given any natural number k, a graph is said to have property J_k if every set of k vertices has a common neighbour in G.

Property J_k is evidently stronger than I_k. It is also clear that every graph with property J_k contains a complete subgraph on $k+1$ vertices. On the other hand, a Kneser-type construction similar to the above one, shows that $f_{J_k}(k+2,n) = o(n)$. That is, there exist K_{k+2} - free graphs with property J_k with small maximum degrees.

It is easily seen that, given a triangle-free graph G, by the addition of some new *edges* creating no triangles we can obtain a graph with property I_2 (i.e. with diameter 2). This statement does not remain true if we require that our graph had property I_3. However, we are unable to decide whether or not the following assertion holds true: every triangle-free graph can be embedded as a subgraph into a triangle-free graph with property I_3. (Here addition of new *vertices* is also permitted.) If the answer to this question is in the affirmative, one can of course then ask how many extra vertices are needed for the embedding. Note that no 4-chromatic graph can be embedded into any triangle-free graph with property I, showing that property I_3 in the above problem cannot be replaced by I.

Another problem which can be raised is the following: given any natural numbers k,r determine the smallest value $g_{I_k}(r) = g$ such that there exists a K_r-free graph with g vertices and property I_k (and containing k independent points). The proof of our Theorem 4 gives that $g_{I_k}(3) \geq 2^{k+1}$.

2 RESULTS AND PROOFS

Theorem 2. Let $f_I(n)$ denote the maximum integer f such that every graph on n vertices having property I has a vertex of degree at least f. Then we have

$$f_I(n) = (1 + o(1))\,\frac{n \text{ loglog } n}{\log n}\,.$$

Proof. The upper bound can be established by a routine calculation, as follows. Let G be a graph with property I and suppose that the maximum degree D is less than n loglog n/log n. Setting $k = [\log n/\log\log n]$, the number m_k of independent k-tuples can be estimated by

$$m_k \geq \frac{n(n-D)(n-2D)\ldots(n-(k-1)D)}{k!} > \frac{(n-kD)^k}{k!}\,.$$

Using the fact that each independent k-tuple has a common neighbour in G, we obtain

$$\frac{n\cdot D}{2} \geq |E(G)| \geq \frac{m_k\cdot k}{2\binom{D-1}{k-1}} > \left(\frac{n}{D} - k\right)^k \frac{D}{2}\,,$$

which yields the desired upper bound.

To prove that our result is sharp we take n points and divide them into t equal classes, where t will be specified later. Each class induces a complete subgraph, whereas every pair of points belonging to different classes will be joined by an edge independently with probability $p = \varepsilon/t$ ($\varepsilon > 0$ is an arbitrarily fixed small constant). Choosing now t points, one from each class, the probability that we cannot find a common neighbour for them is $(1-p^t)^{n-t}$. Thus the probability that our graph has not got property I is at most

$$\left(\frac{n}{t}\right)^t (1-p^t)^{n-t} < e^{t\cdot\log(n/t)\ -\ (n-t)(\varepsilon/t)^t}$$

which tends to zero if $t = [(1-\varepsilon)\log n/\text{loglog } n]$. On the other hand, we almost certainly have that the maximum degree of the vertices does not exceed $(n/t) + (2\varepsilon n/t) < (1+3\varepsilon)n \log\log n/\log n$. $\square$

Carrying out the same random construction a little more carefully, we can also ensure that no large independent sets appear in our graph. In this way we can obtain

Theorem 3. Let $f_I(r,n)$ denote the maximum integer f such that every K_r-free graph on n vertices having property I has a vertex of degree at least f. Then we have

$$f_I(w(n)\log n, n) \leq (2 + o(1)) \frac{n \log\log w(n)}{\log w(n)},$$

where $w(n)$ is an arbitrary function tending to infinity.

Proof. Let $\varepsilon > 0$ be a fixed small constant. Take n distinct points and divide them into t equal classes, where the value of t will be specified later. Define a random graph on these vertices, as follows. Any two points belonging to the same class are joined by an edge independently with probability $q = 1 - (4/w(n))$, while any edge running between different classes will be drawn in with probability $p = \varepsilon/t$. The probability that our graph contains a complete subgraph on $w(n)\log n$ vertices is at most

$$\binom{n}{(w(n)\log n)} \cdot q^{(w(n)\log n)^2/2} < e^{-w(n)\log^2 n}$$

which tends to zero as $n \to \infty$. On the other hand, it is almost certain that our graph will not contain any independent set S of size $\geq t(2+\varepsilon)\log n/\log w(n)$. Otherwise at least $s = [(2+\varepsilon)\log n/\log w(n)]$ elements of S would belong to the same class, and the probability of this event can be bounded (from above) by

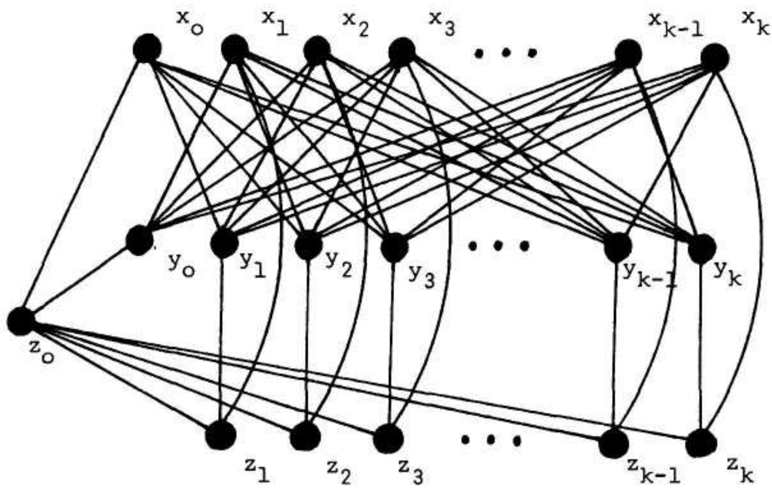
$$t\binom{n/t}{s}\left(\frac{4}{w(n)}\right)^{s^2/2} < e^{\frac{s}{2}(2\log n - s\log(w(n)/4))}$$

which tends to zero again. Further, the probability that our graph does not have property I can be estimated by

$$\leq st\,\binom{n}{st}\,(1-p^{st})^{n-st}$$

and this approaches zero, if $t = [\log w(n)/(2+2\varepsilon)\log\log w(n)]$ and n is large. Finally, it is obvious that the maximum degree of the vertices is almost certainly smaller than $(1+2\varepsilon)n/t$. □

Figure 1



Theorem 4 (stated in §1) is an immediate consequence of Theorem 1 and the following

Lemma. Let G be a triangle-free graph on n vertices. If every set of $\lceil \log n \rceil$ independent vertices has a common neighbour in G, then G has property I.

Proof. Assume, indirectly, that there exists a maximum integer $k < n$ such that G has property I_k but one can find a $k+1$ element independent set $X = \{x_0, x_1, \ldots \ldots, x_k\} \subseteq V(G)$ with no common neighbour in G. Let y_i denote a vertex connected to all points in $X - x_i$; $i = 0,1,\ldots,k$. We obviously have that $x_i y_i \notin E(G)$. Further, let z_o be any point joined to both x_o and y_o and choose a common neighbour z_i to each triple $\{z_o, x_i, y_i\}$; $i = 1, 2,\ldots,k$. (See Fig. 1.) Using the fact that G is triangle-free it follows that all the above defined vertices are different from each other, and

$y_1,\ldots,y_k$, $z_1,\ldots,z_k$ induce a one-factor in G. Thus, for every subset $A \subseteq \{1,2,\ldots,k\}$,

$$H_A := \{y_1 \mid i \in A\} \cup \{z_i \mid i \notin A\}$$

is an independent k-tuple and, by property I_k, we can find a vertex v_A connected to all elements of H_A. All the 2^k v_A's defined in this way are different, showing that $k < \lceil \log n \rceil$, a contradiction. □

References

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