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FAMILIES OF FINITE SETS IN WHICH NO SET IS COVERED BY THE UNION OF r OTHERS[†]

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ABSTRACT

Let $f_r(n,k)$ denote the maximum number of k-subsets of an n-set satisfying the condition in the title. It is proved that

$$f_r(n, r(t-1)+1+d) \leqq \binom{n-d}{t} \Big/ \binom{k-d}{t} \qquad \text{for } n \text{ sufficiently large}$$

whenever $d=0,1$ or $d \leqq r/2t^2$ with equality holding iff there exists a Steiner system $S(t, r(t-1)+1, n-d)$. The determination of $f_r(n,2r)$ led us to a new generalization of BIBD (Definition 2.4). Exponential lower and upper bounds are obtained for the case if we do not put size restrictions on the members of the family.

1. Preliminaries

Let X be an n-element set. For an integer k, $0 \leqq k \leqq n$ we denote by $\binom{X}{k}$ the collection of all the k-subsets of X, while 2^X denotes the power set of X. A family of subsets of X is just a subset of 2^X. It is called *k-uniform* if it is a subset of $\binom{X}{k}$. A Steiner system $\mathcal{S} = S(t,k,n)$ is an $\mathcal{S} \subset \binom{X}{k}$ such that for every $T \in \binom{X}{t}$ there is exactly one $B \in \mathcal{S}$ with $T \subset B$. Obviously, $|\mathcal{S}| = \binom{n}{t}/\binom{k}{t}$ holds. A $\mathcal{P} \subset \binom{X}{k}$ is called a *(t,k,n)-packing* if $|P \cap P'| < t$ holds for every pair $P, P' \in \mathcal{P}$. V. Rödl [10] proved that

$$\text{(1)} \qquad \max\{|\mathcal{P}| : \mathcal{P} \text{ is a } (t,k,n)\text{-packing}\} = (1-o(1))\binom{n}{t} \Big/ \binom{k}{t}$$

holds for all fixed k, t whenever $n \to \infty$.

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Let $\lceil a \rceil$ ($\lfloor b \rfloor$) denote the smallest (greatest) integer (not) exceeding a (b), respectively. We will use the Stirling formula, i.e., $n! \sim (n/e)^n \sqrt{2\pi n}$.

2. Uniform r-cover-free families

We call the family of sets $\mathcal{F}$ *r-cover-free* if $F_0 \not\subset F_1 \cup \cdots \cup F_r$ holds for all $F_0, F_1, \ldots, F_r \in \mathcal{F}$. ($F_i \neq F_j$ if $i \neq j$.) Let us denote by $f_r(n,k)$ the maximum cardinality of an r-cover-free family $\mathcal{F} \subset \binom{X}{k}$, $|X| = n$. Let us set $t = \lceil k/r \rceil$. Then

PROPOSITION 2.1. $\binom{n}{t}/\binom{k}{t}^2 \leqq f_r(n,k) \leqq \binom{n}{t}/\binom{k-1}{t-1}$.

To prove the lower bound we show that there exists a (t,k,n)-packing of this size. A $(t, r(t-1)+1, n)$-packing $\mathcal{P}$ is r-cover-free because $|P \cap P'| \leqq t-1$ holds for all $P, P' \in \mathcal{P}$. Generally

EXAMPLE 2.2. Let $X = Y \cup D$, $|D| = d$, $|Y| = n-d$ and $\mathcal{P}$ a $(t, r(t-1)+1, n-d)$-packing over Y. Define $\mathcal{F} = \{D \cup P : P \in \mathcal{P}\}$.

This example and (1) gives the lower bound in the following theorem.

THEOREM 2.3. *Let $k = r(t-1)+1+d$ where $0 \leqq d < r$. Then for $n > n_0(k)$*

$$(1-o(1))\binom{n-d}{t}\bigg/\binom{k-d}{t} \leqq f_r(n,k) \leqq \binom{n-d}{t}\bigg/\binom{k-d}{t} \tag{2}$$

holds in the following cases:

(a) $d = 0, 1$,

(b) $d < r/(2t^2)$,

(c) $t = 2$ *and* $d < \lceil 2r/3 \rceil$.

Moreover, equality holds in (2) *iff a Steiner-system $S(t, k-d, n-d)$ exists.*

This theorem determines asymptotically $f_r(n,k)$ for several values of r and k. The first uncovered case is $r = 3$, $k = 6$. The obvious conjecture that the maximum $\mathcal{F}$ has the structure given by Example 2.2 is not true (cf. Theorem 2.6). A subset $A \subset F \in \mathcal{F}$ is called an *own subset* of F if $A \not\subset F'$ holds for all $F \neq F' \in \mathcal{F}$.

Let us suppose $X = \{1, 2, \ldots, n\}$ and define $\max F = \max\{i : i \in F\}$.

DEFINITON 2.4. A family $\mathcal{F} \subset \binom{X}{u}$, t, $r \geqq 2$, is called a *near t-packing* if $|F \cap F'| \leqq t$ holds for all distinct $F, F' \in \mathcal{F}$, moreover, $|F \cap F'| = t$ implies $\max F \not\in F'$ (in words: the t-subsets of F containing $\max F$ are own subsets).

PROPOSITION 2.5. *If $\mathcal{F} \subset \binom{X}{u}$ is a near t-packing then $\mathcal{F}$ is r-cover-free.*

PROOF. Suppose $F \subset F_1 \cup \cdots \cup F_r$, $F_i \in \mathcal{F}$. Since $|F \cap F_i| \leqq t$, the sets $F \cap F_i$ form a partition into t-subsets of F. Choose F_i containing max F. Then $F \cap F_i$ is a t-subset of F containing max F and $F \cap F_i \subset F$. However, $F \cap F_i$ was supposed to be an own subset of F, a contradiction. □

THEOREM 2.6. *There exists a near 2-packing* $\mathcal{F} \subset \binom{X}{2r}$ *with* $(n^2/(4r-2)) - o(n^2)$ *edges.*

This theorem and Proposition 2.1 give that $f_r(n, 2r) = (1 + o(1))n^2/(4r-2)$. It is easy to see that

PROPOSITION 2.7. *For fixed k and r,*

$$\lim f_r(n,k) \Big/ \binom{n}{t} = \limsup_{n\to\infty} f_r(n,k) \Big/ \binom{n}{t} = c_r(k)$$

exists whenever $n \to \infty$.

By Proposition 2.1 and (2) we have

$$1 \Big/ \binom{k-d}{t} \leqq c_r(k) \leqq 1 \Big/ \binom{k-1}{t-1}.$$

In Chapter 5 we get the slightly better

$$c_r(k) \leqq (k - dt)/t \binom{k-1}{t-1}$$

but we have no general conjecture for the value of $c_r(k)$ not covered by Theorems 2.3 and 2.6.

3. r-Cover-free families without size restriction

Denote by $f_r(n)$ the maximum cardinality of an r-cover-free family $\mathcal{F} \subset 2^X$, $|X| = n$.

THEOREM 3.1. $(1 + 1/4r^2)^n < f_r(n) < e^{(1+o(1))n/r}$.

REMARK. In the case $r = 1$ the constraints reduce to $F_0 \not\subset F_1$, i.e., the well-known Sperner-property. Hence (see [11])

$$f_1(n) = \binom{n}{\lfloor n/2 \rfloor}.$$

Suppose now that n is not too large compared to r.

EXAMPLE 3.2. Let q be the greatest prime power with $q \leqq \sqrt{n}$. Let $Y = \mathrm{GF}(q) \times \mathrm{GF}(q)$ be the underlying set and consider the graphs of the polynomials of degree at most d over the finite field $\mathrm{GF}(q)$. Set

$$\mathscr{F}_{q,d} = \{\{(x, g(x)) : x \in \mathrm{GF}(q)\} : g(x) = a_0 + a_1 x + \cdots + a_d x^d,\ a_i \in \mathrm{GF}(q)\}.$$

Then $|F \cap F'| \leqq d$ holds for $F, F' \in \mathscr{F}_{q,d}$, thus it is a $\lfloor (q-1)/d \rfloor$-cover-free family.

This yields the lower bound for $2r^2 < n$ in the following:

THEOREM 3.3. *For $r = \varepsilon\sqrt{n}$ we have*

$$(1 - o(1))\sqrt{n}^{\lfloor 1/\varepsilon \rfloor + 1} \leqq f_r(n) \leqq n^{\lceil 2/\varepsilon^2 \rceil}.$$

For $n < \binom{r+2}{2}$ we have the following easy

PROPOSITION 3.4. *If $n < \binom{r+2}{2}$ then $f_r(n) = n$.*

4. Proof of Proposition 2.1

If $\mathscr{F}$ is a maximal (t, k, n)-packing then for every $G \in \binom{X}{k}$ there is an $F \in \mathscr{F}$ such that $|G \cap F| \geqq t$ holds. Hence we have

$$\binom{n}{k} \leqq \sum_{F \in \mathscr{F}} \left| \left\{ G \in \binom{X}{k} : |G \cap F| \geqq t \right\} \right| \leqq |\mathscr{F}| \binom{k}{t} \binom{n-t}{k-t}.$$

Using

$$\binom{n}{k}\binom{k}{t} = \binom{n}{t}\binom{n-t}{k-t},$$

this yields the lower bound.

For the proof of the upper bound let us define the family $\mathscr{N}(F)$ the *non* own parts of F with respect to $\mathscr{F}$, i.e.,

$$\mathscr{N}(F) = \{T \subset F : |T| = t,\ \exists F' \neq F,\ F' \in \mathscr{F},\ T \subset F'\}.$$

LEMMA 4.1. *If $\mathscr{F}$ is an r-cover-free family, $F \in \mathscr{F}$ and $T_1, T_2, \ldots, T_r \in \mathscr{N}(F)$ then $|\bigcup T_i| < k$.*

PROOF. Trivial, choose $F \neq F_i \in \mathscr{F}$ with $T_i \subset F_i$ and note $F \not\subset F_1 \cup \cdots \cup F_r$. □

LEMMA 4.2. $|\mathscr{N}(F)| \leqq \binom{k-1}{t}$.

PROOF. In view of Lemma 4.1 $\mathcal{N}(F)$ fulfills the following conditions: (i) $\mathcal{N}(F)\subset\binom{F}{t}$, $rt\geqq|F|$ and (ii) $A_1\cup\cdots\cup A_r\neq F$ for $A_1,\ldots,A_r\in\mathcal{N}(F)$.
Thus by Lemma 1 (Frankl [8]), $|\mathcal{F}|\leqq\binom{k-1}{t}$ holds. □

Now Lemma 4.2 implies that each $F\in\mathcal{F}$ has at least

$$\binom{k}{t}-\binom{k-1}{t}=\binom{k-1}{t-1}$$

own subsets. Consequently,

$$|\mathcal{F}|\binom{k-1}{t-1}\leqq\binom{n}{t}$$

holds, yielding the desired upper bound.

5. Proof of Theorem 2.3

Let $\mathcal{F}_0=\{F\in\mathcal{F}:\exists S\subset F, |S|\leqq t-1$, such that $S\subset F'\in\mathcal{F}$ implies $F'=F\}$, i.e. $\mathcal{F}_0$ denotes the family of members of $\mathcal{F}$ having an own subset of size smaller than t. Clearly, we have

$$|\mathcal{F}_0|\leqq\binom{n}{t-1}. \tag{3}$$

LEMMA 5.1. *If* $F\in\mathcal{F}-\mathcal{F}_0$ *and* $T_1,T_2,\ldots,T_{d+1}\in\mathcal{N}(F)$ *then* $|\bigcup T_i|<(d+1)t$.

PROOF. Suppose for contradiction that $|\bigcup T_i|=(d+1)t$ and let $\mathcal{P}=\{T_1,T_2,\ldots,T_{d+1},S_1,S_2,\ldots,S_{r-d-1}\}$ be a partition of F such that $|S_i|=t-1$. Then for each $P\in\mathcal{P}$ there exists a $F_P\in\mathcal{F}$, $F_P\neq F$ with $P\subset F$. Hence $F\subset\bigcup\{F_P:P\in\mathcal{P}\}$, a contradiction. □

LEMMA 5.2. *For* $F\in\mathcal{F}-\mathcal{F}_0$ *we have*

$$|\mathcal{N}(F)|\leqq d\binom{k-1}{t-1}, \tag{4}$$

$$|\mathcal{N}(F)|\leqq\binom{k}{t}-\binom{k-d}{t}\qquad \text{if } k>2t^3d, \tag{5}$$

$$|\mathcal{N}(F)|\leqq\binom{k}{2}-\binom{k-d}{2}\qquad \text{if } t=2,\quad k\geqq\tfrac{5}{2}d+2. \tag{6}$$

Moreover, equality holds in (5) *or* (6) *iff* $|\bigcup\{T\in\binom{X}{t}:T\notin\mathcal{N}(F)\}|=k-d$.

PROOF. Let us define $m(k,t,d)=\max\{|\mathcal{N}| : \mathcal{N}\subset\binom{k}{t},\ \mathcal{N}$ does not contain $d+1$ pairwise disjoint members$\}$ where $k>td$, k, t, d are positive integers. Erdös, Ko and Rado [6] proved that

$$m(k,t,1)=\binom{k-1}{t-1} \qquad \text{for } k\geqq 2t$$

and

$$m(k,t,d)\leqq d\binom{k-1}{t-1}$$

was shown by Frankl (cf. [7] or [9]). For $k>k_0(t,d)$ Erdös [3] proved that

$$m(k,t,d)=\binom{k}{t}-\binom{k-d}{t}.$$

Later $k_0(t,d)<2t^3d$ was established by Bollobás, Daykin and Erdös [2]. For $t=2$,

$$m(k,2,d)=\binom{d}{2}+d(k-d)$$

was proved by Erdös and Gallai [5] (for $k\geqq(5d/2)+2$). The uniqueness of the optimal families was proved both in [2] and [5]. These results and Lemma 5.1 imply (4)–(6). □

From now on we suppose that one of the cases (a), (b), or (c) holds, i.e., (5) or (6) is fulfilled. We apply the following theorem of Bollobás [1].

LEMMA 5.3. *Let $A_1,\dots,A_m$ and $B_1,\dots,B_m$ be finite sets and suppose that $A_i\cap B_i=\varnothing$ and $A_i\cap B_j\neq\varnothing$ holds for all $i\neq j$. Then*

$$\sum_i \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}}\leqq 1. \tag{7}$$

Moreover, if $|A_i|=a$, $|B_i|=b$ holds for all i then equality holds in (7) *only if $|\bigcup A_i|=|\bigcup B_i|=a+b$.*

Divide $\mathcal{F}-\mathcal{F}_0$ into two parts: $\mathcal{F}_1=\{F\in\mathcal{F}-\mathcal{F}_0 : |\mathcal{N}(F)|<\binom{k}{t}-\binom{k-d}{t}\}$, $\mathcal{F}_2=\mathcal{F}-\mathcal{F}_0-\mathcal{F}_1$. Then for each $F\in\mathcal{F}_2$ we have a d-subset $D(F)\subset F$, such that $|(F-D(F))\cap F'|\geqq t$ implies $F=F'$.

Now let $T_1,T_2,\dots,T_m$ be the family of *all* minimal own subsets of size at most t of the members of $\mathcal{F}$, i.e., $T_i\subset(F\cap F')$ and $F,F'\in\mathcal{F}$ imply $F=F'$, and for all $x\in T_i$ there exists $F'\neq F$, $F'\in\mathcal{F}$ such that $(T_i-\{x\})\subset F\cap F'$. Define

$$X_i = \begin{cases} X - T_i & \text{if } T_i \subset F \in (\mathscr{F}_0 \cup \mathscr{F}_1), \\ X - T_i - D(F) & \text{if } T_i \subset F \in \mathscr{F}_2. \end{cases}$$

Clearly $X_i \cap T_i = \emptyset$. We claim that $X_i \cap T_j \neq \emptyset$ holds for all $i \neq j$. If $X_i = X - T_i$ then this follows from the minimality of T_i, i.e., $T_j \not\subset T_i$. Suppose $X_i = X - T_i - D(F)$. If T is a t-subset of $T_i \cup D(F)$, then either $T = T_i$ or $T \cap D(F) \neq \emptyset$ holds. Since T_j is an own subset of some $F' \in \mathscr{F}$ and $F \in \mathscr{F}_2$, we infer $T_j \not\subset (T_i \cup D(F))$, i.e., $T_j \cap X_i \neq \emptyset$.

Now Lemma 5.3 yields

$$1 \geqq \sum_{F \in \mathscr{F}} \sum_{\substack{T_i \subset F \\ T_i \text{ min. own} \\ \text{part of } F}} \frac{1}{\binom{|X_i| + |T_i|}{|T_i|}} \geqq \frac{|\mathscr{F}_0|}{\binom{n}{t-1}} + \frac{\binom{k-d}{t}+1}{\binom{n}{t}} |\mathscr{F}_1| + \frac{\binom{k-d}{t}}{\binom{n-d}{t}} |\mathscr{F}_2|.$$

Straightforward calculation shows that if $n > 2dt\binom{k}{t}$, then the coefficient of $|\mathscr{F}_2|$ is the smallest, hence we have

$$|\mathscr{F}_0| + |\mathscr{F}_1| + |\mathscr{F}_2| = |\mathscr{F}| \leqq \binom{n-d}{t} \Big/ \binom{k-d}{t},$$

as desired. Moreover, equality can hold only if $\mathscr{F}_0 = \mathscr{F}_1 = \emptyset$. Finally, to get the extremal family we apply the second part of Lemma 5.3, which yields that each $D(F)$ is the same.

6. Proof of Theorem 2.6

We are going to use probabilistic methods.

LEMMA 6.1. *Let Y be an m-element set, $m \geqq 2r$. Then there exist $(2r-1)$-uniform families $\mathscr{P}_1, \ldots, \mathscr{P}_s$ such that $P \cap P' = \emptyset$ for $P, P' \in \mathscr{P}_i$, $|\mathscr{P}| \geqq (m/(2r-1)) - 12r^2\sqrt{m}$, $|P \cap P'| \leqq 2$ for all $P \in \mathscr{P}_i$, $P' \in \mathscr{P}_j$ and $s > m^{3/2}/r^2$.*

PROOF. Let $A_1, A_2, \ldots, A_u$ be pairwise disjoint $(2r-1)$-element subsets of Y, $u = \lfloor m/(2r-1) \rfloor$. Consider $3s$ permutations chosen independently at random of Y, $\pi_1, \pi_2, \ldots, \pi_{3s}$ where $s = \lceil m^{3/2}/r^2 \rceil$. Define the family $\mathscr{R}_i$ as $\{\pi_i(A_j) : 1 \leqq j \leqq u\}$. To obtain the families $\mathscr{P}_i$ we will delete the "bad" members of $\mathscr{R}_i$.

For $B \in \binom{Y}{3}$ we have that

$$\text{Prob}(B \text{ is covered by some members of } \mathscr{R}_i) = \frac{|\mathscr{R}_i| \binom{2r-1}{3}}{\binom{m}{3}} < \frac{4r^2}{m^2}.$$

Hence we get

$$\mathrm{E}\left(\# B\in\binom{Y}{3}\text{ which are covered by }\mathcal{R}_i\text{ and }\mathcal{R}_j\text{ as well}\right)<\binom{m}{3}\left(\frac{4r^2}{m^2}\right)^2<\frac{8r^4}{2m}.$$

Finally we get

$$\text{(8)}\qquad \mathrm{E}(\# R\in\bigcup\mathcal{R}_i:\text{ there exists }R'\in\bigcup\mathcal{R}_i,\ |R\cap R'|\geqq 3)$$
$$\leqq\binom{3s}{2}\cdot 2\cdot\frac{8r^4}{3m}<3s(8r^2\sqrt{m}).$$

Now, call a permutation π_i "bad" if $\mathcal{R}_i$ contains at least $12r^2\sqrt{m}$ members R with the property $|R\cap R'|\geqq 3$ for some $R'\in\bigcup_{j\neq i}\mathcal{R}_i$. Then by (8) we have

$$\mathrm{E}(\#\text{ bad }\mathcal{R}_i)\leqq 2s.$$

Thus there exists a choice of the random permutations $\pi_1,\ldots,\pi_{3s}$ such that at most $2s$ out of $\mathcal{R}_1,\ldots,\mathcal{R}_{3s}$ are bad. Suppose by symmetry $\mathcal{R}_1,\ldots,\mathcal{R}_s$ are not bad. Each $\mathcal{R}_i$ contains less than $12r^2\sqrt{m}$ members R such that $|R\cap R'|\geqq 3$ for some $R'\in\bigcup_{j\neq i}\mathcal{R}_j$. Let $\mathcal{P}_i$ be the family obtained from $\mathcal{R}_i$ after deleting these R. Then $\mathcal{P}_1,\ldots,\mathcal{P}_s$ satisfy all the requirements. □

Now the construction of the desired $\mathcal{F}\subset\binom{X}{2r}$, where $X=\{1,2,\ldots,n\}$ is the following. Let $X=Y_1\cup Y_2\cup\cdots\cup Y_a\cup Y_0$ where

$$|Y_1|=\cdots=|Y_a|=m=\lceil r^2n^{2/3}\rceil,\quad a=\lfloor n^{1/3}/r^2\rfloor,\quad Y_i\cap Y_j=\varnothing$$

for all $0\leqq i<j\leqq a$. Take a copy of the families defined by Lemma 6.1 for each Y_i, we get $\mathcal{P}_1^i,\mathcal{P}_2^i,\ldots,\mathcal{P}_s^i$. Finally, set $\mathcal{F}=\{P\cup\{j\}:P\in\mathcal{P}_j^i,\ 1\leqq i<j/m\}$. We have

$$|\mathcal{F}|\geqq\sum_{1\leqq j\leqq n}(j/m-1)\left(\frac{m}{2r-1}-12r^2\sqrt{m}\right)\geqq\binom{n}{2}\frac{1}{2r-1}-O(n^{5/3}).$$

7. Proof of Proposition 2.7

Let k and r be fixed. Let $g_r(n,k)$ be the maximum size of an r-cover-free family $\mathcal{F}$ such that for all $F\in\mathcal{F}$, $T\subset F$, $|T|=t-1$ we have an $F'\neq F$, $F'\in\mathcal{F}$ with $(F\cap F')\supset T$.

Such a family $\mathcal{F}$ is called r-cover-free without small own subsets. Deleting successively the members of $\mathcal{F}$ having own $(t-1)$-subsets we can always obtain a $\mathcal{G}\subset\mathcal{F}$, $\mathcal{G}$ is without small own subsets. Obviously,

$$|\mathcal{F}-\mathcal{G}|\leqq\binom{n}{t-1}$$

hence we have

$$f_r(n,k)-\binom{n}{t-1}\leqq g_r(n,k)\leqq f_r(n,k).$$

Hence it is sufficient to prove that for all $\varepsilon>0$ and n there exists an $N_0(n,\varepsilon)$ such that

$$g_r(N,k)\Big/\binom{N}{t}>\left(g_r(n,k)\Big/\binom{n}{t}\right)-\varepsilon \tag{9}$$

holds whenever $N>N_0$.

Let $\mathcal{F}\subset\binom{X}{k}$, $|X|=n$ be an r-cover-free family without own parts of cardinality at most $(t-1)$ such that $|\mathcal{F}|=g_r(n,k)$. By Rödl's theorem (i.e. by (1)) for $N>N_0(n,\varepsilon)$ there exists a (t,n,N)-packing $\mathcal{P}$ over the N-element set Y, with

$$|\mathcal{P}|>(1-\varepsilon)\binom{N}{t}\Big/\binom{n}{t}.$$

Replace each $P\in\mathcal{P}$ by a copy of $\mathcal{F}$. We obtain an r-cover-free family on N points, yielding (9).

8. Proof of Theorem 3.1

The upper bound of 3.1 comes from Proposition 2.1 using the obvious $f_r(n)\leqq\Sigma_k f_r(n,k)$ and the Stirling formula.

The lower bound was obtained from Proposition 2.1, also, with $k=n/4r$. We can get somewhat better lower bounds carrying out the proof given in [4] for the case $r=2$.

9. Proof of Theorem 3.3 and Proposition 3.4

Let $\mathcal{F}\subset 2^X$ be an r-cover-free family and define

$$\mathcal{F}_t=\{F\in\mathcal{F}: F \text{ has own subset of size most } t\}.$$

Clearly, $|\mathcal{F}_t|\leqq\binom{n}{t}$.

LEMMA 9.1. *If $F\in(\mathcal{F}-\mathcal{F}_t)$ and $F_1,F_2,\dots,F_i\in\mathcal{F}$ then*

$$\left|F-\bigcup_{j\leqq i}F_j\right|>t(r-i). \qquad \square$$

This lemma implies that:

$$F_1,\dots,F_{r+1}\in(\mathcal{F}-\mathcal{F}_t) \quad \text{then} \quad \left|\bigcup_{i\leqq r+1}F_i\right|\geqq(r+1)(tr+2)/2. \tag{10}$$

For $r = \varepsilon\sqrt{n}$ and $t = \lceil 2/\varepsilon^2 \rceil$ the right-hand side of (10) is greater than n. Thus $|\mathcal{F} - \mathcal{F}_t| \leqq r$, i.e.,

$$|\mathcal{F}| \leqq \binom{n}{\lceil 2/\varepsilon^2 \rceil} + \varepsilon\sqrt{n} \leqq n^{\lceil 2/r^2 \rceil} \qquad \text{for } t \geqq 2.$$

The case $t = 1$ follows from Proposition 3.4.

To prove Proposition 3.4 we apply induction on n. The statement is trivial, e.g., for $n \leqq r$. Suppose $\mathcal{F} \subset 2^X$, $|X| = n$, $\mathcal{F}$ is r-cover-free. If some $F \in \mathcal{F}$ has a 1-element own subset, say $\{x\}$, then the statement follows by induction, applied to $\mathcal{F} - \{F\}$, $X - \{x\}$. If $\mathcal{F}_1 = \varnothing$, and $|\mathcal{F}| > r$, then (10) implies

$$|X| = n \geqq \binom{r+2}{2},$$

a contradiction. Thus $|\mathcal{F}| \leqq r < n$ holds.

10. Final remarks

The paper is a continuation of the earlier work of the authors [4] where they dealt with the case $r = 2$, i.e., $A_0 \not\subset A_1 \cup A_2$. The above topic is full of problems which are related to designs and error-correcting codes.

Open Problem. Suppose $\mathcal{F} \subset 2^X$, $|X| = n$, $\mathcal{F}$ is r-cover-free, $|\mathcal{F}| > n$. For a given r denote by $n(r)$ the minimum of such n. Then by Proposition 3.4 we have

$$\binom{r+2}{2} \leqq n(r) < r^2 + o(r^2).$$

(The upper bound comes from the example of an affine plane of order at least $r + 1$.) One can prove $n(r) > (1 + o(1))\frac{5}{6}r^2$. We conjecture that $\lim n(r)/r^2 = 1$, or even stronger $n(r) \geqq (r+1)^2$. (We can prove this for $r \leqq 3$.)

Added in proof. Theorem 3.1 was proved independently by Hwang and Sós [12]. They apply the estimations of $f_r(n)$ for group testing.

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