A NOTE ON THE SIZE OF A CHORDAL SUBGRAPH

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1. Let G(n,m) denote an undirected simple graph with n vertices and m edges. A graph is chordal (triangulated, rigid circuit) if every cycle of length > 3 has a chord: namely, an edge joining two nonconsecutive vertices on the cycle. The class of chordal graphs includes trees, k-trees, block graphs, interval graphs and complete graphs. Moreover, chordal graphs are known to be perfect [1] and they possess a number of desirable algorithmic characteristics. Chordal graphs also arise in several application areas: solution of sparse systems of linear equations [12], evolutionary trees [2], facility location [3], and scheduling [11]. Chordal graphs are studied by many, e.g. [5], [6], [9], [10]. If a graph is not chordal, it is quite appropriate to ask the following questions:

(1) What is the maximum order of a chordal subgraph?

(2) What is the maximum size of a chordal subgraph?

In answer to (1) recently Dearing, Shier, and Warner [4] have developed a polynomial time algorithm to generate a maximal chordal subgraph. It may be pointed out that their algorithm does not generate a chordal subgraph of maximum order. In an earlier paper [8], Erdős and Laskar have determined asymptotically the minimum number of edges to be deleted

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from a graph such that the resulting graph is a chordal subgraph of maximum order.

This paper is a first attempt to answer to (2). Let $f(n,t)$ denote the smallest positive integer, for which every $G(n,f(n,t))$ contains a chordal subgraph of size at least t. We show here that, $f(n,n) = [\frac{n^2}{4}] + 1$. Further, we prove that any $G(n,[\frac{n^2}{4}] + 1)$ contains a chordal subgraph of size $n(1+\varepsilon)$, if $n > n_0(\varepsilon)$ where $\varepsilon > 0$ is a fixed positive number. At present we cannot determine the exact value of ε. In fact, in such a graph we show the existence of a tringle xyz, with deg x + deg y + deg z $> n(1+\eta)$ for small $\eta > 0$, so that the triangle xyz, together with the incident edges of x,y,z give such a chordal subgraph. In this connection, it may be pointed out that Edwards [7] has shown that any graph $G(n,m)$ with $m \geq \frac{n^2}{3}$ contains a triangle xyz, where deg x + deg y + deg z $\geq 2n$, and hence $G(n,m)$ contains a chordal subgraph of at least size $2n-3$.

2. Let $f(n,t)$ denote the smallest integer for which every $G(n,f(n,t))$ contains a chordal subgraph of at least t edges. Let $N(v)$ denote the neighbors of v and $N[v] = N(v) \cup \{v\}$.

First we prove the following:

<u>Theorem 1.</u> $$f(n,n) = [\frac{n^2}{4}] + 1.$$

<u>Proof.</u> Suppose G is a graph with n vertices and $[\frac{n^2}{4}] + 1$ edges. It suffices to show that such a graph G always has a chordal subgraph with n edges, and that there exists a graph with n vertices and $[\frac{n^2}{4}]$ edges whose all chordal subgraphs are of size $\leq n-1$.

Let G be a graph with n vertices and $[\frac{n^2}{4}] + 1$ edges. First note that G must have a vertex x with deg $x > \frac{n}{2}$. Let v be a maximum degree vertex and deg $v = \frac{n}{2} + t$, $t > 0$. Now there must exist a vertex $y \in N(v)$ such that, deg $y > \frac{n}{2} - t$. If not, then

$$2|E| = 2\{[\frac{n^2}{4}] + 1\} = \sum_{x \in N[v]} \deg x + \sum_{x \in N[v]} \deg x$$

$$\leq \frac{n}{2} + t + (\frac{n}{2} + t)(\frac{n}{2} - t) + (\frac{n}{2} - t - 1)(\frac{n}{2} + t)$$

$$= 2(\frac{n^2}{4} - t^2).$$

i.e. $|E| \leq \frac{n^2}{4} - t^2$, a contradiction.

Let $y \in N(v)$ such that deg $y > \frac{n}{2} - t$. Now y must be adjacent to at least one other vertex u in N(v); otherwise, G has at least

$|N(y)| + |N(v)| \geq \frac{n}{2} - t - 1 + \frac{n}{2} + t = n + 1$ vertices, a contradiction. Thus we have a $K_3 = \{v,y,u\}$. The edges incident to v,y,u together with K_3 form a chordal subgraph of G with n edges.

To show the existence of a graph G with n vertices and $[\frac{n^2}{4}]$ edges, all of whose chordal subgraphs have $\leq$ n-1 edges, we note that the Turan graph [13] is complete bipartite $K_{[\frac{n}{2}],\{\frac{n}{2}\}}$ with n vertices and $[\frac{n^2}{4}]$ edges and has no triangles. A spanning tree of this graph is a chordal subgraph with maximum number n-1 of edges. □

Our next theorem proves a stronger result.

Theorem 2. Any graph $G(n,[\frac{n^2}{4}] + 1)$ contains a chordal subgraph of at least $n(1+\varepsilon)$ edges if $n > n_0(\varepsilon)$ where $\varepsilon > 0$ is a fixed positive number.

Proof. As in theorem 1, let v be a maximum degree vertex with deg $v = \frac{n}{2} + t$, $t > 0$. Let $y \in N(v)$ with deg $y > \frac{n}{2} - t$.

Suppose $t > \eta n$, for some $\eta > 0$ and $N(v) = \{y_1, y_2, \ldots, y_{\frac{n}{2}+t}\}$, and $\deg y_1 \geq \deg y_2 \geq \ldots \geq \deg y_{\frac{n}{2}+t}$. If $\deg y_1 \leq \frac{n}{2} - t + \eta^2 n$, then

$$\sum_{i=1}^{\frac{n}{2}+t} \deg y_i \leq \left(\frac{n}{2} + t\right)\left(\frac{n}{2} - t + \eta^2 n\right)$$

$$= \frac{n^2}{4} - t^2 + \eta^2 \frac{n^2}{2}$$

$$< \frac{n^2}{4} \quad (\because t > \eta n)$$

Hence,

$$2|E| = \sum_{i=1}^{\frac{n}{2}+t} \deg y_i + \sum_{x \notin N(v)} \deg x$$

$$< \frac{n^2}{4} + \frac{n^2}{4} - t^2$$

$$= \frac{n^2}{2} - t^2$$

Thus,

$$|E| < \frac{n^2}{4} - \frac{t^2}{2}, \text{ a contradiction.}$$

Hence, $\deg y_1 < \frac{n}{2} - t + \eta^2 n$. Then y_1 must be adjacent to $\eta^2 n$ vertices y_i in $N(v)$. Pick any such vertex, say y_r. Then the triangle v, y_1, y_r forms a chordal subgraph with at least $\deg v + \deg y_1 + \deg y_r - 3$ edges, i.e. at least $\frac{n}{2} + t + \frac{n}{2} - t + \eta^2 n + 1 + 2 - 3 = n(1+\eta^2)$ edges, and we have our desired chordal subgraph.

Thus, to complete the proof we have to show that $t > \eta n$ for some $\eta > 0$. As before, we consider the triangle v, y, y_r, where v is a maximum degree vertex with $\deg v = [\frac{n}{2}] + t$ and $y, y_r \in N(v)$ and $\deg y \geq \frac{n}{2} - t + 1$. If $\deg y_r \geq \eta n$, we have a chordal subgraph consisting of the triangle vyy_r, together with the edges incident with v, y, and y_r, having at least $\frac{n}{2} + t + \frac{n}{2} - t + 1 + \eta n = n(1+\eta) - 1$ edges. So assume that $\deg y_r < \eta n$. Delete y_r from G, the resulting graph has n-1 vertices and at least $\frac{n^2}{4} + 1 - \eta n > \frac{(n-1)^2}{4}$ edges. Hence, we can repeat the argument. Suppose we can continue this process for ℓ times. The resulting graph G_1 has then $m = n-\ell$ vertices and $> \frac{m^2}{4}$ edges. Consider the triangle $v'y'y_r'$ in G_1 as of the construction, where v' is a maximum degree vertex with $\deg v' = \frac{m}{2} + t'$ and $\deg y' > \frac{m}{2} - t'$. If $\deg y_r' > \ell + \eta n$, then we have

$$\deg v' + \deg y' + \deg y_r' > \eta n + \ell + \frac{m}{2} + t' + \frac{m}{2} - t' = n + \eta n,$$

and we have our desired chordal subgraph.

If $\deg y_r' \leq \eta n + \ell$, choose ℓ to be very small, say, $\ell = \frac{n}{10}$ (it is large compared to η). Counting the edges of G (note that there are $\frac{n}{10}$ vertices of G whose degrees are $<\frac{n}{10} + \eta n$), we have

$$|E| \leq \frac{1}{2}\left[\left(\frac{n}{2} + t\right)\left(n - \frac{n}{10}\right) + \frac{n}{10}\left(\frac{n}{10} + \eta n\right)\right]$$

Now if $t \leq \eta n$,

$$|E| \leq \frac{1}{2}\left[\left(\frac{n}{2} + \eta n\right)\left(n - \frac{n}{10}\right) + \frac{n}{10}\left(\frac{n}{10} + \eta n\right)\right] < \frac{n^2}{4},$$

a contradiction.

Thus $t > \eta n$, and we complete our proof. □

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