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ON TWO UNCONVENTIONAL NUMBER THEORETIC FUNCTIONS AND ON SOME RELATED PROBLMES

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Let $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, $\omega(n)$=k is the number of distinct prime factors of n. Put

$$f(n) = \sum_{\substack{p|n \\ p^{\alpha} \leq n < p^{\alpha+1}}} p^{\alpha}, \; F(n) = \max \sum_{\substack{a_i \leq n \\ (a_i, a_j)=1}} a_i$$

where all the prime factors of the a's are the prime factors of n. Trivially $f(n) \leq F(n)$ and $f(n) = F(n)$ if $n = p^{\alpha}$ but $f(n) = F(n)$ is possible for arbitrarily large $\omega(n)$, we will give the simple proof later Probably there is no simple characterization of the integers for which $f(n) = F(n)$.

For almost all n we probably have

$$\frac{F(n) - f(n)}{n} \to \infty \qquad \dots (1)$$

and perhaps even for almost all n

$$f(n) = o(n \log \log n), \; F(n) > c\, n \log \log n. \qquad \dots (2)$$

I seem to have difficulties with proving (1) and the second inequality of (2), but it is easy to prove that for almost all integers $F(n)/n \to \infty$.

$f(n)/n$ at first sight almost is a conventional additive function, but this is misleading and in fact $\frac{f(n)}{n}$ does not have a mean value. We have in fact that

$$\lim \sup \frac{1}{x} \sum_{n=1}^{x} \frac{f(n)}{n} = \infty, \; \lim \inf \frac{1}{x} \sum_{n=1}^{x} \frac{f(n)}{n} < \infty \qquad \dots (3)$$

$\frac{f(n)}{n}$ does not have a distribution function but the logarithmic density of the integers n for which $\frac{f(n)}{n} < c$ exists and is a continuous increasing function of c.

Put

$$\max_{n<x} f(n) = m(x), \max_{n<x} F(n) = M(x).$$

I will give a detailed proof of

$$\limsup_{x\to\infty} m(x) \Big/ \frac{x \log x}{\log \log x} = 1. \quad \dots (4)$$

(4) was posed by me at the "Schweitzer competition" in 1982.

I conjecture but can not prove that

$$m(x) = (1+o(1)) \frac{x \log x}{\log \log x}. \quad \dots (5)$$

(5) is of course much stronger than (4).

I can not decide whether for infinitely many x

$$m(x) = M(x). \quad \dots (6)$$

In fact perhaps (6) holds for every sufficiently large x (or even for all x). I feel that (3), (4), (5) and (6) justify the study of these somewhat unconventional functions. Now I give the proof of my theorems, further problems will be stated in the text.

THEOREM 1. *For every* k *there is an* n_k *for which*

$$F(n_k) = n_k, \omega(n_k) = k. \quad \dots (7)$$

Theorem 1 is perhaps surprising but the proof is not difficult. Let t be large and $u_1, u_2, \dots, u_k$ are integers so that the sums $\sum_{i=1}^{k} a_i u_i$, $\Sigma a_i \leq k$, a_i integers are all distinct. Let now x be very large compared to t and the u's and let p_i be the prime nearest to $t^{u_i} x^{1/k}$ and put $n_k = \prod_{i=1}^{k} p_i$. It is easy to see that n_k satisfies (7). To see this observe that the a's (the integers composed of the p's) are either n_k or are greater than n_k and thus can not be used as summands for $F(n_k)$ or are $\leq (1+o(1))\, n_k/t$ thus their total contribution would be less than

$$(1+o(1))\frac{2^k n}{t} = o(n)$$

which completes the proof of Theorem 1.

It would be of some interest to have some more precise condition which implies $F(n) = n$ and to get a reasonably good estimation for the number of integers $n \leq x$ which satisfy $F(n) = n$.

THEOREM 2. *For every k there is an m_k for which*

$$F(m_k) = f(m_k)\,,\ \omega(m_k) = k. \qquad \ldots\ (8)$$

Let $p_1, p_2, \ldots, p_k$ be any set of k primes, x be a large integer for which for every i, $i = 1, \ldots, k$

$$\frac{2}{3}x < p_i^{\alpha_i} < x\,. \qquad \ldots\ (9)$$

It follows from elementary theorems on diophantine approximation that such an x exist. Let m be the smallest integer greater than x of the form $\prod_{i=1}^{k} p_i^{\beta_i}$, $\beta_i > 0$. It again follows from elementary theorems on diophantine approximation that $m_k < x\,(1+\varepsilon)$. Clearly (8) is satisfied for m_k since $\frac{m_k}{2} < p_i^{\alpha_i} < m_k$ thus $p_i^{\alpha_i} + p_j^{\alpha_j} > m_k$ and hence in the definition of $F(m_k)$ the a's must be powers of primes, which completes the proof of Theorem 2. Here too, it would be of some interest to estimate the number of integers $m < x$ satisfying (8) and to obtain good conditions which imply (8). ((9) clearly implies (8)).

In fact it is not hard to prove that there are integers n with $\omega(n) = (1+o(1))$ $\log n/_{\log\log n}$ for which (8) holds. The proof of (6) will easily give this too.

Are there integers $n \neq p^{\alpha}$ for which (7) and (8) both hold, i.e., $F(n) = f(n) = n$? I doubt that such integers exist. In fact, are there integers n for which $n \neq p^{\alpha}$ and $f(n) = n$?

Let $2, 3, \ldots, p_k$ — the sequence of consecutive primes. Denote by x_k the smallest integer for which there is an exponent α_i, $i = 1, 2, \ldots, k$ satisfying

$$\frac{x_k}{2} < p_i^{\alpha_i} < x_k. \qquad \ldots\ (10)$$

It easily follows from the box principle that

$$x_k < \exp\exp k^{1+\varepsilon}. \qquad \ldots\ (11)$$

Unfortunately I have no lower bound for x_k.

THEOREM 3. *$F(n)/_n \to \infty$ if one neglects a sequence of density 0.*

Let $k > k_0(\varepsilon, n)$. It is well known that the density of integers n which have more than $(1-\eta)\log\log k$ prime factors $\leq k$ is greater than $1-\varepsilon$. This follows easily by the method of Turàn*. Let now $p.q|n$, $p < q \leq k$. It is easy to see that if $n > n_0(k)$ then there are integers α and β with

* See e.g. P.D.T.A. Elliott, Probabilistic Number Theory Springer Verlag 1980.

$$(1-\varepsilon)\, n < p^{\alpha} q^{\beta} < n. \qquad \text{(12)}$$

Thus from (12) we obtain that for these n $F(n) > \frac{1}{3} \log \log k$ which implies Theorem 3.

I hoped that by this method I can prove that for almost all n $F(n) > c \log \log n$ and in fact that for almost all n $F(n)=\left(\frac{1}{2}+o(1)\right) \log \log n$ and I hope that I will be successful in this, but there are some difficulties with diophantine properties of powers of primes. One of the difficulties is the following: Let p and q be two primes, $a_1 < a_2 < \dots$ are the integers composed of p and q. Define $f(p, q, \varepsilon)$ as the smallest integer so that for every $y \geq f(p,q,\varepsilon)$ there is an a_i satisfying $y < a_i < y(1+\varepsilon)$. I have no satisfactory upper or lower bounds for $f(p, q, \varepsilon)$. To obtain such an estimation may be genuinely difficult but perhaps one can get around this difficulty and obtain $F(n)=\left(\frac{1}{2}+o(1)\right) \log \log n$, but I have not yet succeeded in doing this.

THEOREM 4.

$$\limsup_{x\to\infty} m(x) \,.\, (x \log x (\log \log x)^{-1})^{-1} = 1.$$

Let $2 < 3 < \dots < p_k$ be the primes not exceeding $\log y (\log \log y)^4$. By the prime number theorem or a more elementary theorem we have $k > c_1 \log y (\log \log y)^3$. For each i, $i=1,2, \dots, k$ let β_i be the least exponent for which $p_i^{\beta_i} \geq y$. Clearly

$$y \leq p_i^{\beta_i} < y \log y (\log \log y)^4.$$

Now by a simple computation we see that there is a z, $y < z < y \log y (\log \log y)^4$ for which there are more than $c_2 \log y \log \log y$ of the numbers $p_i^{\beta_i}$ satisfying

$$z < p_i^{\beta_i} < z\left(1+\frac{1}{\log \log y}\right). \qquad \text{(13)}$$

We obtain (13) by a simple counting argument using $k > c \log y (\log \log y)^3$. Let now $s=[(1-\varepsilon) \log y (\log \log y)^{-1}]$ and let $p_1, p_2, \dots, p_s$ be any set of s primes satisfying (13). Clearly

$$T_s = \prod_{i=1}^{s} p_i < y^{1-\varepsilon/2}. \qquad \text{(14)}$$

Let n be the smallest multiple of T_s which is greater than $z\left(1+\frac{1}{\log \log y}\right)$. From (13) and (14) we have

$$n < z\left(1+\frac{2}{\log\log y}\right). \qquad \dots (15)$$

On the other hand we have by (13)

$$f(n) > s\,z > (1-\varepsilon)\,\frac{\log y}{\log\log y}\,z > (1-2\varepsilon)\,\frac{\log z}{\log\log z}\,.\,z\,. \qquad \dots (16)$$

(15) and (16) clearly prove Theorem 4.

Denote $\max_{n\leq x}\ \omega(n)=h(x)=(1+o(1))\,\frac{\log x}{\log\log x}$. Clearly

$$m(x) < x\,h(x).$$

Theorem 4 can also be stated as

$$\limsup_{x\to\infty} m(x)\,/_{x\,h(x)} = 1\,.$$

I am sure that

$$\lim_{x\to\infty}\frac{x\,h(x)-m(x)}{x} = \infty\,. \qquad \dots (17)$$

I could not prove (17) and (5), but perhaps I overlook a simple idea.

THEOREM 5. *lim inf* $f(n)/_{n^{2/3}} = 2$.

Let p be a large prime, $n=pq$ where q is the greatest prime less than p^2. Clearly $f(n) = p^2+q = (2+o(1))n^{2/3}$. Let p_1 and p_2 be the two smallest prime factors of n. A simple argument shows that the contribution of p_1 and p_2 to $f(n)$ is $\geqslant (2-o(1))n^{2/3}$, which completes the proof of Theorem 5.

THEOREM 6.

$$\limsup\ \frac{1}{x}\sum_{n=1}^{x} f_{(n)}/n = \infty\ .$$

Let $2, 3, \dots, p_k$ be the first k primes. It easily follows from elementary theorems on diophantine approximations that for every $\varepsilon > 0$ and k there are arbitrarily large values of $x = x(\varepsilon, k)$ so that for every i, $1 \leqslant i \leqslant k$ there is an α_1 for which

$$(1-\varepsilon)x < p_i^{\alpha_i} < x. \qquad \dots (18)$$

As stated in the proof of Theorem 3 for every $y > x$ all but $(1-\varepsilon)y$ integers $n < y$ have more than $(1-n)\log\log k$ prime factors $\geqslant p_k$. Choose $y = 2x$, then we obtain from (18)

$$\sum_{n=x}^{2x} \frac{f(n)}{n} > \frac{1}{4} x \log \log k$$

which proves Theorem 6.

Using (10) and (11) we can prove that for infinitely many x

$$\sum_{n=1}^{x} f_{(n)}/n > c\, x \log \log \log \log x. \qquad \ldots \quad (19)$$

In the opposite direction we prove

THEOREM 7.

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} < c\, x \log \log \log x.$$

To prove Theorem 7 it will suffice to show that

$$\sum_{n=x}^{2x} \frac{f(n)}{n} < c_1\, x \log \log \log x. \qquad \ldots \quad (20)$$

I think that (19) is closer to the "truth" than (20).

To prove (20) we interchange the order of summation and obtain

$$\left(p^{\alpha_p(x)} \leqslant 2x < p^{\alpha_p(x)+1}\right)$$

$$\sum_{n=x}^{2x} \frac{f(n)}{n} < \frac{1}{x} \sum_{p} \frac{2x}{p}\, p^{\alpha_p(x)} = 2 \sum \frac{p^{\alpha_p(x)}}{p} = \sum\nolimits_1 + \sum\nolimits_2 \qquad \ldots \quad (21)$$

where in Σ_1 $p \leqslant (\log x)^{10}$ and in Σ_2 $(\log x)^{10} < p \leqslant x$.

Clearly

$$\sum\nolimits_1 < 4x \sum_{p<(\log x)^{10}} \frac{1}{p} < c_2\, x \log \log \log x. \qquad \ldots \quad (22)$$

Further write

$$\Sigma_2 = \Sigma_2' + \Sigma_2'' \qquad \dots (23)$$

where in Σ_2' $p^{\alpha_p(x)} < \dfrac{x}{\log \log x}$ and in Σ_2''

$\dfrac{x}{\log \log x} < p^{\alpha_p(x)} \leq 2x$, in both Σ_2' and Σ_2'' we have $p > (\log x)^{10}$. Clearly

$$\Sigma_2' < \frac{2x}{\log \log x} \sum_{p<x} \frac{1}{p} < c_3\, x. \qquad \dots (24)$$

Thus to complete our proof we only have to estimate Σ_2''. We evidently have

$$\sum\nolimits_2'' < \sum\nolimits_r \sum_p \frac{2x}{p} \qquad \dots (25)$$

where in the inner sum

$$\left(\frac{x}{\log \log x}\right)^{1/r} < p < x^1/3 \qquad \dots (26)$$

and r runs from 2 to $\dfrac{\log x}{\log \log x}$. The upper bound for r is given by the fact that if

$r > \dfrac{\log x}{10 \log \log x}$ then $x^{1/r} < (\log\ x)^{10}$ and in Σ_2'' we have $p > (\log x)^{10}$

Now by Brun's method we easily obtain that (I supress here some details which can easily be filled in by the interested reader, we use that the number of primes in y, $y+t$ is $< ct(\log t)$

$$\sum_{p \text{ in } (26)} \frac{1}{p} < c_4 \ \frac{\log \log x}{\log x} \ . \qquad \dots (27)$$

Thus by (25) and (27) and $r < 10 \dfrac{\log x}{\log \log x}$

$$\sum\nolimits_2'' < c_4 x \,. \qquad \dots (28)$$

Thus finally by (21), (22), (23), (24) and (28) we obtain (20) which completes the proof of Theorem 7.

THEOREM 8.

$$\textit{lim inf}\ \frac{1}{x} \sum_{n=1}^{x} \frac{f(n)}{n} < \infty\ .$$

Theorem 9. *The logarithmic density of the integers n for which $f(n)\,/_n < c$ exists.*

In other words

$$\frac{1}{\log x} \sum_{\substack{n<x \\ f(n)<cn}} \frac{1}{n} \to f(c) .$$

and $f(c)$ is a continuous function of c. I do not prove Theorem 8 and 9 here since my proof at the moment is probably too complicated. The reason for the truth of Theorems 8 and 9 is that the contribution of the large primes p to $f(n)$ is on the average (logarithmic average) small, but I hope to return to some of these questions at an other occasion.

In a problem of mine (Nieuw Archief voor Wiskunde (1983) p. 81, problem 623 I ask the following question :

A set $S=\{a_1, \dots, a_l\}$ is admissible for n if $a_i \leq n$, $1 \leq i \leq l$, and $(a_i, a_j)=1$ if $i \neq j$. Define

$$G(n)= \max \sum a_i \qquad \dots \quad (29)$$

where the summation is extended over all admissible sets S.

Van Lint and I prove that

$$G(n)=\sum_{p\leq n} p+(1+o(1))\, n\, \pi(n^{1/2}) . \qquad \dots \quad (30)$$

Is there a relatively simple algorithm for computing $G(n)$? Is it true that for $n>n_0(r)$ at least one of the a 's in (29) must have more than r prime factors (compare Theorem 1) ? This is easy for $r=1$ in fact by a simple computation (which I have not carried out) one could determine the largest n for which all the a 's in (29) are powers of primes. Observe that an a_i can occur in (29) only if $F(a_i)=a_i$. Put $H(n)= \sum_{p\leq n} p+n\, \pi\, (n^{1/2})$. Our proof with van Lint in fact gives

$$H(n)-n^{3/2-c} < G(n) < H(n) .$$

It is not entirely trivial to prove that

$$\lim \frac{H(n)-G(n)}{n} = \infty$$

and if we make plausible (but hopeless) assumptions about the distribution of primes then for every $\varepsilon > 0$ and $n > n_0(\varepsilon)$

$$G(n) > H(n) - n^{1+\varepsilon}.$$

Put

$$f^{(*)}(n)=\sum_{\substack{p\mid n \\ p^{\alpha}\leq n<p^{\alpha+1}}} p^{\alpha-1}$$

$f^{*}(n)$ is a slight modification of $f(n)$, but $f^{*}(n)$ behaves much more like an ordinary additive function. First of all $f^{*}(n)/_{n}$ has a distribution function

$$\frac{1}{x}\sum_{n<x}\frac{f^{(*)}(n)}{n}=c\ ,\ \limsup f^{*}(n)/_{n}=\infty$$

but it is not yet clear to me how fast $f^{*}(n)$ can tend to infinity. Also it is not hard to prove that the density of integers for which $f(n+1)>f(n)$ as $\frac{1}{2}$ and the same holds for $F(n)$ and $f^{*}(n)$.

Put finally

$$f^{**}(n)=\sum_{\substack{p\mid n \\ p^{\alpha}\leq n<p^{\alpha-2}}} p^{\alpha+1}$$

$f^{**}(n)$ no longer causes serious difficulties. Elementary results on diophantine approximation (see (9)) give

$$\limsup f^{**}(n)=\sum_{p}\frac{1}{p^{2}},$$

References

(4) was posed by me at the Schweitzer competition in 1982, Mat. Lapok, 31, p. 198 / in Hungarian / .

See e.g. P.D.T.A. Elliott, 1980 : Probabilistic number theory, *Springer Verlag.*

See e.g. H. Halberstam and K. F. Roth 1983 : Sequences, chapter IV *Springer Verlag,*

Nieuw Archief voor Wiskunde, 1983, p. 81, problem 623.

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