On the Chessmaster Problem

P. Erdős

R.L. Hemminger

D.A. Holton

B.D. McKay

To illustrate the use of the Pigeonhole Principle, Brualdi [1,pp 16,22] considers the following problem. A chessmaster who has 11 weeks to prepare for a tournament decides to play at least one game every day, but in order not to tire himself he decides not to play more than 12 games during any one week. Show that, for any k with $1 \le k < 21$, there corresponds a succession of days during which the chessmaster will have played *exactly* k games.

If we let a_i be the number of games played during the first i days, (and take $a_0=0$) then the above problem translates into the following:

CMP: If $1 \le k \le 21$ and $0=a_1<a_1<a_2<\ldots<a_{77}$ such that $a_{i+7} \le a_i+12 \ \forall i (0 \le i \le 70)$, then $\exists i,j (0 \le i < j \le 77)$ such that $a_j=a_i+k$.

An alternate interpretation could be obtained by replacing $(0 \le i \le 70)$ with $(i=0,7,14,\ldots,70)$ but the generalization we will make seems to be much more chaotic with this interpretation.

It turns out that the implication in CMP holds for all k with $1 \le k \le 77$ but does not hold for any $k>77$ (just take $a_i=i, \forall i$). This moved one of the authors to raise the following general question.

GCMP: Let k,n,b and c be positive integers with $b \le n,c$ and let $A=\{a_i\}_{i=0}^{n}$ be a sequence of integers. Consider the followig conditions on A:

(1) $0=a_0<a_1<a_2<\ldots<a_n$,

(2) $a_{i+b} \le a_i+c, \ \forall i (0 \le i \le n-b)$ and

(3) $\exists i,j \ (0 \le i < j \le n)$ such that $a_j=a_i+k$.

For what values of k,n,b and c does $(1) \wedge (2) \Rightarrow (3)$?

The case $n=b$, $k \le 2b-c-1$ is an Olympiad problem (see [3]) which has been analyzed for other values of k by Hutchinson and Trow [2]; also see [4] for an answer to a question they raised. We have shown that (1) $\wedge$ (2) $\wedge$ $(n=b) \Rightarrow$ (3) precisely for those k with $k\lfloor b/k\rfloor > c-b$.

We have already noted that the implication does not hold for $k>n$ so hereafter we will assume that $k\le n$. The just-stated results suggest that $c=2b$ is somehow limiting. The following examples confirm that impression since (1) $\wedge$ (2) hold in them, but (3) fails to hold.

EXAMPLE 1: For $c>2b$ and k even, define A by taking

$$a_i=\begin{cases} 2i & \text{if } \lfloor 2i/k\rfloor \text{ is even} \\ 2i+1 & \text{otherwise}\end{cases}.$$

EXAMPLE 2: For $c\ge 2b$ and k odd; indeed for k/d odd where d is any divisor of (k,b), the g.c.d. of k and b, define A by taking

$$a_{md+r}=2md+r \quad (0\le r\le d-1,\ 0\le md+r\le n).$$

We observe that for $i<j$ in Example 1, a_j-a_i is one of the numbers $2(j-i)$, $2(j-i)-1$ or $2(j-i)+1$, and so $a_{i+b}\le a_i+(2b+1)\le a_i+c$. On the other hand one easily sees that (3) fails to hold since k is even.

In Example 2 we note that $a_{md+r}+k=2md+r+hd$ (with h odd)$=(2m+h)d+r$ which is not in A since $2m+h$ is odd. Moreover, noting that $a_t=2t-r_t$ where r_t denote the remainder when t is divided by d, we see that $a_{t+b}-a_t=2(t+b)-r_t$ where r_t denotes the remainder when t is divided by d, we see that $a_{t+b}-a_t=2(t+b)-r_{t+b}-(2t-r_t)=r_t-r_{t+b}+2b=2b$ since d divides b. Thus $a_{t+b}=a_b+2b\le a_t+c$.

The following result is useful in attempting a proof by contradiction.

LEMMA 1: If (1) and the negation of (3) hold, then $a_{i+k}\ge a_i+2k, \forall i (0\le i\le n-k)$.

PROOF: From the negation of (3), we see that for each i, the numbers $a_{i+1},a_{i+2},\ldots,a_n$ include at most one each from each of the $k-1$ sets $\{a_i+1,a_i+k+1\}$, $\{a_i+2,a_i+k+2\}$, . . . , $\{a_i+k-1,a_i+2k-1\}$. Likewise none of them equals a_i+k and so the lemma follows by (1) and the Pigeonhole Principle.

The following two results, due to the first author, confirms most of the conjectures put forward by the other three authors at the Silver Jubilee conference. As in Lemma 1, the Pigeonhole Principle is a vital ingredient.

LEMMA 2: If $c \le 2b-1, 1 \le k \le n$ and r is the number such that $a_r < k \le a_{r+1}$, then $r+b \le n \Rightarrow$ (3) holds.

PROOF: Since $a_0 = 0 < k$ and $a_k \ge k$ by (1), such an r exists. But then the $2b+1$ numbers $a_0+k, a_1+k, \ldots, a_b+k, a_{r+1}, a_{r+2}, a_{r+b}$ all lie in the set $\{k, k+1, \ldots, k+c\}$ by (2), and hence (by the Pigeonole Principle) two of them are equal since $c+1 \le 2b$.

COROLLARY. If $c \le n+1$, then (3) holds.

PROOF. If $r+b > n$, then we have

$$\begin{aligned} a_k &\le a_n-(n-k) \text{ by (1)} \\ &\le a_{n-b}+c-(n-k) \text{ by (2)} \\ &< a_r+c-(n-k) \text{ since } r > n-b \\ &\le a_r+c-(c-1-k) \text{ since } c \le n+1 \\ &= a_r+k+1 \end{aligned}$$

Thus $a_k < 2k$ since $a_r < k$. The corollary follows from the two lemmas.

We note in passing that this establishes the claim that the CMP is true for all k with $1 \le k \le 77$.

The situation for $n < c-1$ is confusing at best. For example, for $n=k=c-2=2b-3$, there are always examples satisfying (1) and (2) but not (3); in the particular case $b=5, c=9, n=k=7$,

$$0,3,5,6,8,9,11,14$$

is such a sequence. In fact, computer generated information suggested that the Corollary was sharp on some intervals of values of k; for example, we conjectured that for $c-1=2b-2$, there are k-sequences satisfying (1) and (2) but not (3) for each k with $c-\lfloor b/2 \rfloor \le k < c-1$. We note that this is indeed correct and is consequence of (f) in the Theorem to follow since $c-\lfloor b/2 \rfloor = 2b-1-\lfloor b/2 \rfloor > 3b/2$.

Some semblance of order was restored when we discovered a generalization of the Corollary that gave sharp results. It is (f) in the Theorem to follow. However, in using the computer we shifted emphasis slightly and we need a difinition to explain the shift.

DEFINITION: We call an n-sequence *acceptable* if it satisfies (1) and (2). Moreover, for $c \ge b$, we define the integer $N(b,c,k)$ by $N(b,c,k) = \min\{n \mid (3) \text{ holds for all acceptable } n\text{-sequences}\}$. Thus, for each $n \ge N(b,c,k)$, (3) holds for every acceptable n-sequence, while, for each $n < N(b,c,k)$, there exist an acceptable n-sequence for which (3) fails. Of course we always have $N(b,c,k) \ge b$ (otherwise (2) holds vacously and acceptable N-sequences are easily constructed for which (3) fails) and $N(b,c,k)\geq k$ (otherwise $a_i=i$ gives an acceptable N-sequence for which (3) fails). In fact, we can characterize when equality holds in these two statements as well as when $N(b,c,k)=\infty$. We now summarize these and other results on $N(b,c,k)$. As most of these results are due to two of the authors who feel they are on the verge of even better results we will not include proofs here. However, we should note that (f), which generalizes the Corollary, admits a proof similar to that of the Corollary.

THEOREM: We have the following (recall that $c\geq b$).

(a) If $c>2b$, then $N(b,c,k)=\infty$ for all k.

(b) If $c=2b$ and $2^m|k=>2^m|b$, then $N(b,c,k)=\infty$ for all k.

(c) If $c=2b$ and there is a positive integer m such that $2^m|k$ but $2^m \nmid b$, then $N(b,c,k)\leq b+k-(b,k)$.

(d) If $c<2b$, then $N(b,c,k)\leq b+k-(b,k)$.

(e) If $c<2b$ and $k\leq b$, then $N(b,c,k)=b$ if and only if $k\lfloor b/k\rfloor>c-b$.

(f) If $c<2b$, then $N(b,c,k)=k$ if and only if either $k=b$ or $k\geq \max(b,2(c-b))$.

(g) If $c<2b$, then $N(b,c,c-b)=\max(b,2(c-b))$.

(h) For each positive integer m $N(mb,mc,mk)\geq mN(b,c,k)$.

The computer generated information was very helpful in guessing what was to be proved in several of the above. Values of $N(b,c,k)$ were generated in the case of $c=2b$ for all k, $b\leq 100$ and in the case $c=2b-i$, $1\leq i\leq 10$ for k and b with $1\leq b\leq 50, 1\leq k\leq 2b$. Based on that information we propose the following conjectures and questions.

CONJECTURES AND QUESTIONS:

(i) Equality holds in (c) above. We do know that equality often does not hold in (d).

(ii) We suggest the following generalization of the Corollary (and of (f) in the case of $c=2b-1$).

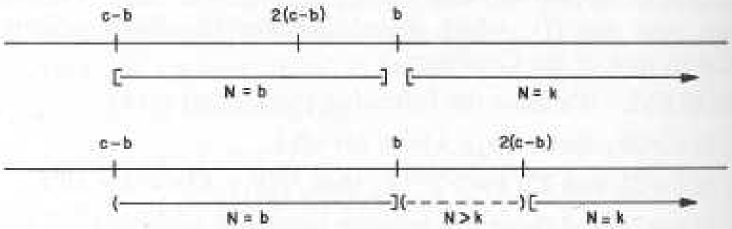
$$N(b,2b-1,k)=mk \text{ if } b=mk,$$

$$N(b,2b-1,k)=mk \text{ for } m\geq 2 \text{ if } \left\lfloor\frac{2b+2m-4}{2m-1}\right\rfloor\leq k<\left\lfloor\frac{2b+2m-6}{2m-3}\right\rfloor,$$

$$N(b,2b-1,k)>mk \text{ otherwise}.$$

(iii) Equality holds in (h).

(iv) Between (e) and (f) of the Theorem, we have quite a bit of information. Some of it is illustrated in the two figures below which correspond to the cases $2(c-b)\le b$ and $2(c-b)>b$. Note that $c-b<b$ since we are assuming we have $c<2b$. The endpoint case $k=c-b$ is covered by (g).



(v) If $c<2b$, then $N(b+k,c+2k,k)=N(b,c,k)+k$.

(vi) In case (b) of the Theorem, the only known infinite acceptable sequences which fail (3) have a periodic stucture. Are there any which are *non-periodic* ?

In order that the reader can get some feel for these results and conjectures the table to follow gives the computer generated information for the case $b=18, c=35$. Each entry is of the form $k{:}N(b,c,k)$.

1:18	2:18	3:18	4:20	5:20	6:18	7:21	8:24	9:18	10:22
11:23	12:24	13:26	14:28	15:30	16:32	17:34	18:18	19:24	20:26
21:27	22:29	23:28	24:30	25:32	26:34	27:34	28:34	29:34	30:34
31:34	32:34	33:34	34:34	35:35	36:36	37:37	38:38	39:39	40:40

and $k{:}k$ for $k\ge 2(c-b)=34$.

References

[1] R.A. Brualdi, *Introductory Combinatorics*, New York, Elsevier North-Holland, Inc., 1977.

[2] J.P. Hutchinson and P.B. Trow, Some pigeonhold principle results extended, *Amer. Math. Monthly* 87 (1980), 648-651. MR82g:0516.

[3] D.O. Shklarsk, N.N. Chentzov and I.M. Yaglom, *The USSR Olympiad Problem Book*, W.H. Freeman, San Francisco, 1962.

[4] W.D. Wei, *Math. Reviews* (July 1982) #82g:0516.