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ON UNAVOIDABLE GRAPHS

F. R. K. CHUNG and P. ERDŐS

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How many edges can be in a graph which is forced to be contained in every graph on n vertices and e edges? In this paper we obtain bounds which are in many cases asymptotically best possible.

I. Introduction

A well-known theorem of Turán [10,11] asserts that every graph on n vertices and e edges must contain a complete subgraph on m vertices if

$$e > \frac{(m-2)}{2(m-1)}(n^2-r^2)+\binom{r}{2}$$

where r satisfies $r \equiv n \pmod{m-1}$ and $1 \leqq r \leqq m-1$.

In this paper we consider a related extremal problem. A graph which is forced to be contained in every graph on n vertices and e edges is called an (n, e)-unavoidable graph, or in short, an unavoidable graph. Let $f(n, e)$ denote the largest integer m with the property that there exists an (n, e)-unavoidable graph on m edges. In this paper we prove the following:

(i) $$f(n,e) = 1 \quad \text{if} \quad e \leqq \left[\frac{n}{2}\right].$$

(ii) $$f(n,e) = 2 \quad \text{if} \quad \left[\frac{n}{2}\right] < e \leqq n.$$

(iii) $$f(n,e) = \left(\frac{e}{n}\right)^2 + O\left(\frac{e^3}{n^{10/3}}\right) \quad \text{if} \quad n \leqq e \leqq n^{4/3}.$$

($O(X)$ denotes a quantity within a constant ratio of X.)

(iv) $$c_1 \frac{\sqrt{e}\log n}{\log\left(\binom{n}{2}/e\right)} < f(n,e) < c_2 \frac{\sqrt{e}\log n}{\log\left(\binom{n}{2}/e\right)}$$

AMS subject classification (1980): 05 C 35 for some constants c_1 and c_2 where $cn^{4/3} < e < \binom{n}{2} - n^{1+c'}$ and c is between 0 and 1.

In particular, we have the following:

(v) $$f(n,e) > (1+o(1))\sqrt{2e} \quad \text{if} \quad e \gg n^{4/3} \quad \left(\text{i.e., } \frac{n^{4/3}}{e} = o(1)\right).$$

(vi)

$$f(n,e) = (1+o(1))\sqrt{2e}\log n / \log\left(\binom{n}{2}/e\right) + O(\sqrt{e}) \quad \text{if} \quad n^{4/3} \ll e = o(n^2).$$

The unavoidable graphs we use for proving (i), (ii) and (iii) are forests which are disjoint unions of stars. In proving (iv), (v) and (vi), we use unavoidable graphs which are disjoint unions of bipartite subgraphs.

II. Preliminaries

We first prove several auxiliary facts:

Lemma 1. *If H is a graph on p vertices and q edges with the property that H is contained in every graph on n points and e edges, then we have*

$$q < \frac{\log n}{\log\left(\binom{n}{2}/e\right)} p.$$

Proof. There are at most n^p ways to map $V(H)$ into $\{1, 2, \ldots, n\}$. Therefore there are at most

$$n^p \binom{\binom{n}{2}-q}{e-q}$$

graphs on n vertices and e edges which contain H. Since there are $\binom{\binom{n}{2}}{e}$ graphs on n vertices and e edges each of which contains H, we have

$$n^p \binom{\binom{n}{2}-q}{e-q} \geq \binom{\binom{n}{2}}{e}.$$

This implies that $n^p > \binom{n}{2}^q / e^q$ and $p \log n / \log\left(\binom{n}{2}/e\right) > q$. ∎

The next lemma is an immediate consequence of Lemma 1.

Lemma 2. *If H is a graph on p vertices and q edges with the property that H is contained in every graph on n points and $n^{2-\varepsilon}$ edges, then we have $q < \frac{p}{\varepsilon}$.* ∎

The following two lemmas can be found in [9].

Lemma 3. *Suppose G is a bipartite graph on e edges such that $V(G)=V_1\cup V_2$ where $|V_1|=m$, $|V_2|=n$ and $E(G)\subseteq V_1\times V_2$. Then G contains a complete bipartite subgraph $K_{a,b}$ with vertex set $U_1\cup U_2$, $|U_1|=a$, $|U_2|=b$, $U_i\subseteq V_i$, $i=1, 2$, if $m\binom{e/m}{b}\geqq a\binom{m}{b}$.* ∎

Lemma 4. *G contains a subgraph isomorphic to $K_{s,t}$ if $n\binom{2e/n}{s}\geqq t\binom{n}{s}$.* ∎

Lemma 5. *Suppose a graph G has n vertices and e edges, with $n^{4/3}\ll e$. Suppose t satisfies $n/\sqrt{e}\ll t\ll e/n$. Then G contains at least $s\geqq(1+o(1))\sqrt{2e}/t$ vertex-disjoint copies of stars S_t.*

Proof. Suppose s is the maximum number of S_t embedded in G. Suppose ε is an arbitrary positive value and $s<(1-\varepsilon)\sqrt{2e}/t$. Let X_i denote the set of vertices in G that the i-th copy of S_t is mapped onto. Let Z denote $V(G)-\bigcup_{i=1}^{s}X_i$. There is at most one vertex in X_i, for each i, adjacent to $2t$ vertices (or more) of Z because of the maximality of s. Therefore the number of edges between X_i and Z is at most $n+2t^2$. The number of edges in the induced subgraph of G on Z is at most $tn/2$. There are at most $\binom{s(t+1)}{2}$ edges in the induced subgraph on $\bigcup_{i=1}^{s}X_i$.

We then have

$$e\leqq s(n+2t^2)+\frac{tn}{2}+\frac{s^2(t+1)^2}{2}\leqq\frac{(s+1)^2(t+1)^2}{2}+\varepsilon e$$

since $s\leqq(1-\varepsilon)\sqrt{2e}/t\ll\frac{e}{n}$ and $t\ll\frac{e}{n}$. This implies $st>(1-\varepsilon)\sqrt{2e}$ which contradicts $s\leqq(1-\varepsilon)\sqrt{2e}/t$. ∎

III. On $f(n, e)$ for $e<n^{4/3}$

The value of $f(n, e)$, for $e<n$, can be easily found by the following observations:

Observation 1. $f(n, e)\geqq 1$ if $e\geqq 1$.

Observation 2. For $e\leqq\left[\frac{n}{2}\right]$, the largest common subgraph of S_e (a star with e leaves) and eP_2 (e independent edges) is a single edge.

Observation 3. If $e>\left[\frac{n}{2}\right]$, a graph on n points and e edges contains P_3.

Observation 4. For $\left[\frac{n}{2}\right]<e<n$, the largest common subgraph of S_e and P_{e+1} (a path on e edges) contains two edges.

Therefore we have

Theorem 1. $f(n, e)=1$ if $e\leqq\left[\frac{n}{2}\right]$ and $f(n, e)=2$ if $\left[\frac{n}{2}\right]<e<n$.
The unavoidable graphs in both cases are paths.

Theorem 2. *Let ε denote a small positive value. Suppose $e<\varepsilon n^{4/3}$. Then any graph on n vertices and e edges contains a disjoint union F of stars, $S_t, S_{t-2}, \ldots, S_{t-2i}, \ldots$ where $t=\left[(1-\varepsilon)\frac{2e}{n}\right]$.*

Proof. Let $F_{t,k}$ denote the forest consisting of $S_t, S_{t-2}, \ldots, S_{t-2k}$. We will prove by induction on k that $F_{t,k}$ can be embedded in any graph G with n vertices and e edges. It is easily seen that $F_{t,1}$ is contained in G. Suppose $F_{t,k-1}$ can be embedded in G. Let $\bar{V}(S_{t-2j})$, $1\leqq j\leqq k-1$, denote the set of vertices in G onto which S_{t-2j} of $F_{t,k-1}$ is embedded. Let U denote the union of $\bar{V}(S_{t-2j})$ for $1\leqq j\leqq k-1$ and $U'=V(G)-U$. If there are two vertices in $\bar{V}(S_{t-2j})$ each of which is adjacent to more than $2t$ vertices in U', then $F_{t,k}$ can be embedded in G. We may assume there is at most one vertex in $\bar{V}(S_{t-2j})$ being adjacent to $\geqq 2t$ vertices in U'. The number of edges in the induced subgraph of G on U is at most $k^2(t-k+1)^2/2$, since $|U|\leqq|V(F_{t,k-1})|\leqq(t-k+1)k$.

The number of edges incident to vertices in U is at most

$$\frac{1}{2}k^2(t-k+1)^2+kn+2tk(t-k+1).$$

Thus the number of edges in the induced subgraph G' of G on U' is at least

$$e-\frac{1}{2}k^2(t-k+1)^2-kn-2tk(t-k+1)\geqq(1-\varepsilon)e-kn,$$

since $\frac{1}{2}k^2(t-k+1)^2+2tk(t-k+1)\leqq\varepsilon e$. Therefore there exists a vertex in G' having degree at least $2((1-\varepsilon)e-kn)/n\geqq t-2k$. This implies that $F_{t,k}$ is contained in G. ∎

Theorem 3. *If $e\leqq n^{4/3}$, we have* $f(n,e)=\left(\frac{e}{n}\right)^2+O\left(\frac{e^3}{n^{10/3}}\right)$.

Proof. Suppose H is an (n,e)-unavoidable graph. For a subset S of vertices in H, we define $N(S)$ to be the number of edges of H incident to vertices in S.

Claim 1. For all S with $|S|=i<\frac{e}{n}-3$, we have $N(S)\leqq\sum_{j\leqq i}\left(\frac{2e}{n}-2j+6\right)$.

Proof of Claim 1. Suppose the contrary. Let k denote the smallest integer so that $N(S)\leqq\sum_{j\leqq i}\left(\frac{2e}{n}-2j+6\right)$ for all S with $|S|=i<k<\frac{e}{n}-3$ and there exists a set $S'\subseteq V(H)$ with $|S'|=k$ and $N(S')>\sum_{j\leqq k}\left(\frac{2e}{n}-2j+6\right)$.

We now consider a graph G satisfying the following conditions.

(α) G has n vertices and e edges.

(β) $V(G)$ can be partitioned into two parts V_1 and V_2 where V_1 has $k-1$ vertices for $k<\frac{e}{3}-3$.

(γ) Every vertex in V_1 is adjacent to every vertex in V_2.

(δ) The induced subgraph on V_2 has maximum degree at most $\frac{2e}{n}-2k+3$.

The existence of such G can easily be seen by noting that

$$\frac{2(e-(k-1)(n-k+1))}{n-k+1} < \frac{2e}{n} - 2k + 3,$$

for n large and $k < \frac{e}{n} - 3$.

Since H is an unavoidable graph, H can be embedded in G. Let $\overline{V}_1$ denote the set of vertices in H which are embedded in V_1. Define $X = \overline{V}_1 \cap S'$, $X' = \overline{V}_1 - S'$ and $Y = S' - \overline{V}_1$. Also we define $N(Y, \overline{V}_1)$ to be the number of edges in H which are incident to vertices in Y and not incident to vertices in $\overline{V}_1$. Let G' be the induced subgraph of H on $X' \cup Y$. G' has at most $2(k-x)$ vertices. From Lemma 1 we know that G' has at most $3(k-x)$ edges since $e < n^{4/3}$.

We then have

$$\begin{aligned} N(S') &\leqq N(X) + |E(G')| + N(Y, \overline{V}_1) \\ &\leqq \sum_{j \leqq x} \left(\frac{2e}{n} - 2j + 6\right) + 3(k-x) + (k-x)\left(\frac{2e}{n} - 2k + 3\right) \\ &\leqq \sum_{j \leqq x} \left(\frac{2e}{n} - 2j + 6\right) + (k-x)\left(\frac{2e}{n} - 2k + 6\right). \end{aligned}$$

On the other hand, we have $N(S') > \sum_{j \leqq k} \left(\frac{2e}{n} - 2j + 6\right)$. This implies

$$(k-x)\left(\frac{2e}{n} - 2k + 6\right) > \sum_{x < j \leqq k} \left(\frac{2e}{n} - 2j + 6\right),$$

which is impossible since $x \leqq k-1$. ∎

Claim 2. $|E(H)| \leqq \left(\frac{e}{n}\right)^2 + 7\left(\frac{e}{n}\right)$.

Proof of Claim 2. We now consider the graph G satisfying the following properties:

(α) G has n vertices and at least e edges.

(β) $V(G)$ can be partitioned into two parts V_1 and V_2 where V_1 has $\left[\frac{e}{n}\right] + 1$ vertices.

(γ) Every vertex in V_1 is adjacent to every vertex in V_2.

(δ) No edge is contained in the induced graph of G on V_1 or V_2.

$$G \text{ has } \left(\left[\frac{e}{n}\right] + 1\right)\left(n - \left[\frac{e}{n}\right] - 1\right) \geqq e + n - \left(\left(\frac{e}{n}\right) + 1\right)^2 \geqq e \text{ edges.}$$

Note that all edges in G are incident to vertices in V_1. Since H is an unavoidable graph, H can be embedded in G. Thus $|E(H)| \leqq N(Y)$ where Y is the set of vertices in V_1 that H is mapped onto. Thus $|Y| \leqq \left[\frac{e}{n}\right] + 2 = s + 2$. From Claim 1 we have

$$|E(H)| \leqq \sum_{j \leqq s+2} \left(\frac{2e}{n} - 2j + 6\right) \leqq \left(\frac{e}{n}\right)^2 + 7\left(\frac{e}{n}\right). \quad ∎$$

We note that Claims 1 and 2 imply

$$f(n, e) \leq \left(\frac{e}{n}\right)^2 + O\left(\frac{e}{n}\right).$$

On the other hand, the graph F as mentioned in Theorem 2 is an unavoidable graph and F contains at least $\left(\frac{e}{n}\right)^2 + O\left(\frac{e^2}{n^{10/3}}\right)$ edges. We then conclude that $f(n, e) = \left(\frac{e}{n}\right)^2 + O\left(\frac{e^2}{n^{10/3}}\right)$ and Theorem 3 is proved. ∎

We note that Theorem 3 is an improved version of a result in [8] which says that for $e < n^{4/3}$ the disjoint union of $\left[\frac{e}{3n}\right]$ copies of stars $S_{\lfloor e/n \rfloor}$ is an unavoidable graph. In Theorem 3 we obtain the best asymptotical value for $f(n, e)$, $e \ll n^{4/3}$.

IV. On $f(n, e)$, for $n^{4/3} < e = o(n^2)$

We want to show the following:

Theorem 4. *For* $n^{4/3} \ll e = o(n^2)$ *we have*

$$f(n, e) = (1+o(1))\sqrt{2e}\,\frac{\log n}{\log\left(\binom{n}{2}/e\right)} + O(\sqrt{e}).$$

We remark that the first term is relevant only if $e > n^{2-o(1)}$ where $o(1)$ denotes a quantity which approaches 0 as n tends to infinity.

Proof. Since the complete graph on $\lceil\sqrt{2e}\,\rceil$ vertices has at least e edges, the unavoidable graph H has at most $\lceil\sqrt{2e}\,\rceil$ (nontrivial) vertices. From Lemma 1, we have

$$|E(H)| \leq \lceil\sqrt{2e}\,\rceil \frac{\log n}{\log\left(\binom{n}{2}/e\right)}.$$

Now we want to establish a lower bound of $f(n, e)$ by finding suitable unavoidable graphs. From Lemma 5 we know that Theorem 4 is true if $k = \log n/\log\left(\binom{n}{2}\Big/e\right)$ is bounded. We only have to consider the situation that k is large.

Suppose ε is an arbitrary positive value. Let x denote the maximum number with the property that G contains x disjoint copies of $K_{s,t}$ where $s = (1-\varepsilon)k$, $t = kn^2/e$. Let V_i denote the set of vertices in G onto which the i^{th} copy of $K_{s,t}$ is embedded. Let U denote $V(G) - \bigcup_{i=1}^{x} V_i$. Because of the maximality of x, the induced subgraph of G on U does not contain $K_{s,t}$. If $x \geq \sqrt{2e}/(s+t)$, then, by $s = o(t)$, we have

$$f(n, e) \geq xst \geq (1+o(1))\sqrt{2e}\,s \geq (1+o(1))\log n/\log\left(\binom{n}{2}/e\right).$$

Therefore, the induced subgraph of G on U has at most $e' < \varepsilon e$ edges for any $\varepsilon > 0$, since, by Lemma 4, we have

$$n\binom{2e'/n}{s} < t\binom{n}{s} \quad \text{and}$$

$$e' \leq \binom{n}{2} n^{-(1+\varepsilon)/k} < \varepsilon e$$

for any $\varepsilon > 0$ and $e < \varepsilon^{1/\varepsilon}\binom{n}{2}$.

For each i, the bipartite graph B_i on V_i and U does not contain two disjoint copies of $K_{s,t}$. Let e_i denote the number of edges between B_i and U. If B_i does not contain $K_{s,2t}$, then by Lemma 3 we have $m\binom{2e_i/m}{s} < 2t\binom{2t}{s}$, where $m = |U| = n - x(s+t) > n - \sqrt{2e}$. This implies that $e_i < 10kn$. Suppose B_i contains one copy of $K_{s,2t}$, then by deleting the s vertices in V_i the remaining graph does not contain $K_{s,2t}$ and has no more than $10kn$ edges. Thus, B_i contains at most $11kn$ edges. Therefore, the number of edges between $\bigcup_{i=1}^{x} V_i$ and U is at most $\frac{e^{3/2}}{n^2 k}\cdot 11kn < \varepsilon e$, since $x \leq \sqrt{2e}/t$.

The induced subgraph on $\bigcup_{i=1}^{x} V_i$ must contain at least $(1-2\varepsilon)e$ edges. Thus $\bigcup_{i=1}^{x} V_i$ contains at least $(\sqrt{2}-3\varepsilon)\sqrt{e}$ vertices. We then have

$$x \geq \frac{(\sqrt{2}-3\varepsilon)\sqrt{e}}{kn^2/e} > \frac{(\sqrt{2}-3\varepsilon)e^{3/2}}{kn^2}.$$

Thus, we have proved that

$$f(n, e) > xst \geq (\sqrt{2}-6\varepsilon)\sqrt{e}\cdot k$$

for any $\varepsilon > 0$ and n sufficiently large. ∎

V. On $f(n, e)$ for $\binom{n}{2} - e = o(n^2)$

When e is close to $\binom{n}{2}$, i.e. $\binom{n}{2} - e = o(n^2)$, our results on $f(n, e)$ are somewhat less precise.

Theorem 5. *Let c and c' denote values satisfying $0 < c, c' < 1$. If $cn^2 < e < \binom{n}{2} - n^{1+c'}$, then we have*

$$c_1 \frac{\sqrt{e}\log n}{\log\left(\binom{n}{2}/e\right)} < f(n, e) < c_2 \frac{\sqrt{e}\log n}{\log\left(\binom{n}{2}/e\right)}$$

for some constants c_1 and c_2.

Proof. The upper bound follows from Lemma 1. Let G denote a graph on n vertices and e edges. Suppose $e=dn^2$, $c<d<1/2$. It can be easily checked from Lemma 4 that any $G'\subseteq G$ of $e/2$ edges contains a complete bipartite graph on two sets of $d'\log n$ vertices where $d'=1/2\log(1/c)$. Now, consider a maximal number of x disjoint copies of $K_{t,t}$, $t=(d'/2)\log n$, which can be embedded in G. By removing $dn/2$ vertices, there are at least $e/2$ edges left, therefore we have $xd'\log n\geqq dn/2$. Thus $dn/(2d'\log n)$ copies of $K_{t,t}$ form an unavoidable graph which has $(dd'n\log n)/8$ edges, and we establish (iv) for $e=dn^2$, where $c<d<1/2$.

Suppose $e=\binom{n}{2}-e'$ and $o(n^2)=e'>n^{1+c'}$. We have

$$\frac{\log n}{\log\left(\binom{n}{2}/e\right)}=(1+o(1))\frac{\binom{n}{2}\log n}{e'}=k$$

Then G contains $K_{s,s}$ with $s=c'k/2$ since

$$\frac{n\binom{2e/n}{s}}{s\binom{n}{s}}\geqq\frac{e'}{n\log n}\left(1-\frac{e'}{\binom{n}{2}}\right)^s$$

$$\geqq\frac{e'}{n\log n}\exp\left[-\frac{e'}{\binom{n}{2}}\,\frac{c'\binom{n}{2}\log n}{2e'}\right]$$

$$\geqq\frac{e'}{n^{1+c'/2}\log n}>1.$$

It is easy to see that G contains $n/10s$ copies of $K_{s,s}$ since deleting $n/5$ vertices in G there are still at least $e/2$ edges left. This proves that $f(n,e)\geqq c_1kn$ for some constant c_1 and the proof of Theorem 5 is completed. ∎

Corollary. *Suppose* $e=\binom{n}{2}-n^{1+c}$. *Then*

$$c_1n^{2-c}\log n<f(n,e)<c_2n^{2-c_1}\log n.$$

Now by combining Theorems 4 and 5 together with the fact that [8] any graph on n vertices and e edges, where $e>n^{4/3}$, contains the forest consisting of $\left\lfloor\frac{n}{3\sqrt{e}}\right\rfloor$ disjoint copies of stars $S_{[e/n]}$, we conclude the following:

Theorem 6. *If c is a value between 0 and 1, and $n^{4/3}<e<\binom{n}{2}-n^{1+c}$, then*

$$c_1\frac{\sqrt{e}\log n}{\log\left(\binom{n}{2}/e\right)}<f(n,e)<c_2\frac{\sqrt{e}\log n}{\log\left(\binom{n}{2}/e\right)}$$

for some constants c_1 and c_2.

VI. Concluding remarks

The unavoidable graph problems are in fact the complementary problem of the universal graphs. For a given calss $\mathbf{H}$ of graphs, a graph G is said to be $\mathbf{H}$-universal if G contains every graph in $\mathbf{H}$ as a subgraph. Univegsal graphs are investigated in [2—7]. Now suppose G is an (n, e)-unavoidable graph. Then the complement G' of G must contain all graphs on $\binom{n}{2}-e$ edges. Furthermore, the complement of a maximum (n, e)-unavoidable graph must be a minimum universal graph on n vertices which contains all graphs on n vertices and $\binom{n}{2}-e$ edges. Thus we have

$$\text{(vii)}\qquad f(n,e)=\binom{n}{2}-g\left(n,\binom{n}{2}-e\right)$$

where $g(n, e')$ denotes the minimum number of edges in a graph on n vertices which contains all graphs on n vertices and e' edges. It is proved in [1] that

$$c_1\frac{n^2}{\log^2 n}\leqq g(n,c_2n)\leqq c_3\frac{n^2\log\log n}{\log n}$$

for some constant c_1, c_2 and c_3. Therefore we have

$$\binom{n}{2}-c_3\frac{n^2\log\log n}{\log n}<f\left(n,\binom{n}{2}-c_2n\right)<\binom{n}{2}-c_1\frac{n^2}{\log^2 n}.$$

For large $e'>n$, $g(n, e')$ approaches $\binom{n}{2}$ rapidly. For example, $g(n, n\log n)$ is at least cn^2 for some constant c (see [1]). Therefore, the relation (vii) does not give interesting bounds for $f(n, e)$ for $e<\binom{n}{2}-cn\log n$.

For $\binom{n}{2}>e>(1-\varepsilon)\binom{n}{2}$, ε small, the value of $f(n, e)$ is still not determined. It would also be of interest to tighten the bound in the case of $n^{4/3}<e<n^{2-\varepsilon}$.

In another direction, one can ask the same questions for r-uniform hypergraphs. Here, the answers are harder to obtain and are known with less precision. This topic will be treated in a later paper.

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F. R. K. Chung

Bell Laboratories
Murray Hill, N. J. 07974, U.S.A.

P. Erdős

Mathematical Institute of the
Hungarian Academy of Sciences
1053 Budapest, Reáltanoda 13—15
Hungary