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ARITHMETICAL PROPERTIES OF PERMUTATIONS OF INTEGERS

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For the finite case let $a_1, a_2, \ldots, a_n$ be a permutation of the integers $1, 2, \ldots, n$ and for the infinite case let $a_1, a_2, \ldots, a_i, \ldots$ be a permutation of all positive integers.

Some problems and results concerning such permutations and related questions can be found in [2] (see in particular p. 94). In [3] the density of the sums $a_i + a_{i+1}$ is estimated from several points of view.

In the present paper we shall investigate the least common multiple and the greatest common divisor of two subsequent elements. First we deal with the least common multiple. For the identical permutation we have $[a_i, a_{i+1}] = i(i+1)$. We show that for suitable other permutations this value becomes considerably smaller.

First we consider the finite case

THEOREM 1. *We have*

$$\min \max_{1 \leq i \leq n-1} [a_i, a_{i+1}] = (1+o(1)) \frac{n^2}{4 \log n} \tag{1}$$

where the minimum is to be taken for all permutations $a_1, a_2, \ldots, a_n$.

One might think that the main reason for not being able to get a smaller value lies in the presence of the large primes (see also the proof). Theorem 2 shows that this is only partly true.

THEOREM 2. *Omit arbitrarily $g(n) = o(n)$ numbers from $1, 2, \ldots, n$ and form a permutation of the remaining ones.*

Then for any fix $\varepsilon > 0$, and n large enough we have

$$\min \max_{1 \leq i \leq n-g(n)-1} [a_i, a_{i+1}] > n^{2-\varepsilon}. \tag{2}$$

On the other hand, for any $\varepsilon(n) \to 0$ we have with a suitable $g(n) = o(n)$

$$\min \max_{1 \leq i \leq n-g(n)-1} [a_i, a_{i+1}] < n^{2-\varepsilon(n)}. \tag{3}$$

An equivalent form of Theorem 2 is: $\dfrac{\log \{\min \max_{1 \leq i \leq n-g(n)-1} [a_i, a_{i+1}]\}}{\log n}$ must tend to 2 for any $g(n) = o(n)$, but it can do this from below arbitrarily slowly for suitable $g(n) = o(n)$.

In the infinite case we obtain a much smaller upper bound:

THEOREM 3. *We can construct an infinite permutation satisfying*

$$[a_i, a_{i+1}] < ie^{c\sqrt{\log i}\log\log i} \tag{4}$$

for all i.

In the opposite direction we can prove only a very poor result:

THEOREM 4. *For any permutation*

$$\limsup_i \frac{[a_i, a_{i+1}]}{i} \geqq \frac{1}{1-\log 2} \sim 3{,}26. \tag{5}$$

Very probably this lim sup must be infinite, and one can expect an even sharper rate of growth.

Concerning the greatest common divisor only the infinite case is interesting.

THEOREM 5. *We can construct an infinite permutation satisfying*

$$(a_i, a_{i+1}) > \frac{1}{2} i \tag{6}$$

for all i.

On the other hand, for any permutation

$$\liminf_i \frac{(a_i, a_{i+1})}{i} \leqq \frac{61}{90}. \tag{7}$$

The right value is probably $\frac{1}{2}$, but we could not yet prove this.

Proofs

PROOF OF THEOREM 1. First we show that any permutation must contain an a_i for which

$$[a_i, a_{i+1}] \geqq (1+o(1))\frac{n^2}{4\log n}.$$

Consider the primes between $\frac{n}{2}$ and n, the number of these is about $\frac{n}{2\log n}$. Hence at least one of them has a left neighbour $\geqq (1+o(1))\frac{n}{2\log n}$, and thus the least common multiple here is $\geqq (1+o(1))\frac{n}{2\log n}\cdot\frac{n}{2}$.

Now we construct a permutation satisfying

$$[a_i, a_{i+1}] \leqq \{1+o(1)\}\frac{n^2}{4\log n} \tag{8}$$

for all $i \leqq n-1$.

The idea is to take the multiples of a prime p as a block, and to separate the blocks by "small" numbers. Then the l.c.m. will not be too large at the border of the blocks. And inside a block

$$[a_i, a_{i+1}] \leqq \frac{n^2}{p} \tag{9}$$

which is good if p is not too small. Finally we have to arrange the numbers having only small prime factors.

Let us see the details. For the primes p up to n let k_p be the minimal exponent for which $q_p = p^{k_p} \geqq 4 \log n$ (i.e. $q_p = p$ if $p \geqq 4 \log n$).

We now define the set S of the "small" separator numbers: take $\Pi(n) - \Pi(\sqrt{n})$ numbers from 1 to some L just leaving out the values q_p and $2q_p$. Obviously $L = (1+o(1)) \frac{n}{\log n}$.

We start the permutation by writing down alternately the primes between n and $\frac{n}{2}$ in decreasing order and the first elements of S in increasing order. (Here a block consists of p alone.) To show (8) we observe that when we arrive to $p \sim cn$, then we have used up $\Pi(n) - \Pi(cn) \sim (1-c) \frac{n}{\log n}$ small numbers, i.e. the l.c.m. of p and its neighbour is

$$c(1-c) \frac{n^2}{\log n} \leqq 4 \log n. \tag{10}$$

For the primes between $\frac{n}{2}$ and $\sqrt{n}$ we slightly improve the construction. We take the largest prime, insert all its multiples (up to n) after it, leaving its double to the end. Now we choose the next even number of S as separator, start the next block with the double of the next prime, put in all the multiples and terminate it by the prime itself. Then we insert the next odd number of S as separator and repeat the alogirthm. (9) and (10) show that (8) is satisfied. We note that for $p < \frac{n}{8}$ we do not have to be so careful about the parity of the separator number, and for $p < \frac{n}{4\sqrt{\log n}}$ we do not really need separators at all.

Next we proceed similarly with the q_p values between $\sqrt{n}$ and $4 \log n$, but here of course we take only those multiples of q_p which have not yet been used up (either in the blocks, or as separators). q_p and $2q_p$ lie at the two ends of a block (they were excluded from S to be now at disposal), hence we can either omit the separators, or put in arbitrarily large numbers as separators. We shall insert as separators the numbers still left, i.e. which have all their prime power factors less than $4 \log n$ (and which were not in S). There are at most

$$2^{\Pi^*(4 \log n)} \sim n^{\frac{c}{\log \log n}}$$

Such numbers, where $\Pi^*(x)$ denotes the number of prime-powers up to x since there are $\Pi(\sqrt{n}) > n^{1/2-\varepsilon}$ blocks for $4 \log n \leqq q_p \leqq \sqrt{n}$, we can consume as separators all the numbers left. We have obviously

$$[a_i, a_{i+1}] \leqq \begin{cases} 2q_p n \leqq 2n^{3/2} & \text{at the border of the blocks} \\ \dfrac{n^2}{q_p} \leqq \dfrac{n^2}{4\log n} & \text{inside.} \end{cases}$$

PROOF OF THEOREM 2. To prove (2) we observe the well-known fact that there are cn numbers up to n which have a prime factor greater than $n^{1-\varepsilon/2}$ $\left(c=(1+o(1))\log\dfrac{1}{1-\varepsilon/2}\right)$, hence we must keep nearly all of them. When we jump from a multiple of a large prime to a multiple of another large prime, then we either jump directly, but then the l.c.m. is at least $(n^{1-\varepsilon/2})^2$, or we insert a small number as separator, but then we need at least $(1+o(1))\dfrac{n}{\log n}$ separators, and so we obtain a l.c.m. greater than $n^{1-\varepsilon/2}\dfrac{n}{\log n}$.

To prove (3) we keep only those numbers whose largest prime factor lies between $n^{\varepsilon(n)}$ and $n^{1-\varepsilon(n)}$. It is well known that we omitted just $o(n)$ numbers (see e.g. [1]). We start the permutation by the largest prime left and its multiples, then we put the next prime followed by its multiples, etc. Here

$$[a_i, a_{i+1}] \leqq \begin{cases} \dfrac{n^2}{p} \leqq n^{2-\varepsilon(n)}, & \text{for two multiples of the same } p \\ n\cdot n^{1-\varepsilon(n)}, & \text{when jumping to a next prime.} \end{cases}$$

PROOF OF THEOREM 3. First we note that it is enough to construct a permutation $a_1, a_2, \ldots,$ of a *subsequence* of the natural numbers which satisfies (4), since we can insert the remaining elements afterwards arbitrarily rarely into this permutation.

We shall use the (probably well-known and nearly trivial) statement of the following lemma:

LEMMA. *Let H be a finite set, $|H|=h$ and $t\leqq h$. Then we can order the subsets having exactly t elements so that $|H_i\cap H_{i+1}|=t-1$ holds for all i.*

PROOF OF THE LEMMA. We prove by induction on h. The initial step is obvious. Now assume that the assertion is true for $h-1$ and for all t. Consider now h and any t. We fix an element x_0, take first all subsets containing x_0 and then take the other ones. Both parts can be ordered suitably by the induction hypothesis for $h-1$, $t-1$, and for $h-1$, t, resp. We have no difficulty either at joining the two parts, since if a "good" order exists, then a simple bijection of H can transform it into another "good" order with a prescribed first (or last) subset.

The construction of the permutation runs by an iterative process. Assume that for some n and $k=n^{\log n}$ we have $a_1, a_2, \ldots, a_k$ ready and no one of them has a prime factor greater than $\dfrac{n}{2}$. We take now all primes between $\dfrac{n}{2}$ and n, and form all the products consisting of v such (distinct) primes where $v=\log n+4\log\log n$. By the lemma we can arrange these products so that any two subsequent terms should differ only in one prime factor. This arrangement will be the next segment of the permutation from a_{k+2}. For a transition element a_{k+1} we can take e.g. any prime between $\frac{n}{2}$ and n. For $i \geqq k+1$ clearly

$$[a_i, {}_{i+1}] \leqq n^{v+1} \sim ke^{2\sqrt{\log k}\log\log k} \leqq ie^{2\sqrt{\log i}\log\log i}.$$

We have formed about

$$r = \binom{\dfrac{n}{2\log n}}{\log n + 4\log\log n} \tag{11}$$

new terms of the permutation thus we arrived at least to a_r.

The algorithm will work if

$$r > (2n)^{\log 2n} \tag{12}$$

holds. Estimating the binomial coefficient in (11) as a power of the smallest factor in the numerator and the greatest factor in the denominator we obtain

$$r \geqq \left(\frac{\dfrac{n}{2\log n} - \log n - 4\log\log n}{\log n + 4\log\log n}\right)^{\log n + 4\log\log n} \geqq \left(\frac{n}{\log^3 n}\right)^{\log n + 4\log\log n}$$

and (12) follows by an easy calculation.

Proof of Theorem 4. First we give a very simple proof of weaker form of (5) with $\frac{3}{2}$ instead of $\frac{1}{1-\log 2}$, i.e. that no permutation can satisfy

$$[a_i, a_{i+1}] < \left(\frac{3}{2} - \varepsilon\right) i \quad \text{with a fix} \quad \varepsilon \quad \text{for} \quad i \geqq i_0. \tag{13}$$

We use the inequality

$$\frac{1}{[a_i, a_{i+1}]} \leqq \frac{1}{3}\left\{\frac{1}{a_i} + \frac{1}{a_{i+1}}\right\} \tag{14}$$

which is equivalent to

$$3 \leqq \frac{a_i}{(a_i, a_{i+1})} + \frac{a_{i+1}}{(a_i, a_{i+1})} \tag{15}$$

and hence it is obvious, since the minimal value of the two terms on the right-hand side of (15) is 1 and 2.

Assuming (13) we obtain

$$\sum_{i=1}^{n} \frac{1}{[a_i, a_{i+1}]} \geqq \sum_{i=i_0}^{n} \frac{1}{[a_i, a_{i+1}]} > \left(\frac{2}{3} + \varepsilon'\right) \sum_{i=i_0}^{n} \frac{1}{i} \geqq \left(\frac{2}{3} + \varepsilon'\right) \log n - K.$$

On the other hand, using (14) we have

$$\sum_{i=1}^{n} \frac{1}{[a_i, a_{i+1}]} \leqq \frac{1}{3}\left\{\sum_{i=1}^{n} \frac{1}{a_i} + \frac{1}{a_{i+1}}\right\} < \frac{2}{3} \sum_{i=1}^{n} \frac{1}{i} \leqq \frac{2}{3}\log n + K'$$

which is a contradiction if n is large enough.

Now we turn to the proof of (5). Assume indirectly that for some permutation, $\varepsilon>0$ and i_0 we have

$$[a_i, a_{i+1}] < i\frac{1}{1-\log 2+\varepsilon} \quad \text{if} \quad i \geqq i_0. \tag{16}$$

This clearly implies also

$$a_i < i\frac{1}{1-\log 2+\varepsilon} \quad \text{for} \quad i \geqq i_0$$

hence $a_1, a_2, \ldots, a_n$ are all smaller than

$$N = n\frac{1}{1-\log 2-\varepsilon} \tag{17}$$

if n is large enough. From now on we shall consider only the a_i-s with $i \leqq n$.

Let us call the primes greater than $\sqrt{N}$ and smaller then N "large primes". If a_i and a_{i+1} have different large prime factors, then $[a_i, a_{i+1}] \geqq N$ in contradiction to (16). Hence we must insert "separators" between a_i-s containing different large prime factors (the separators cannot have large prime factors, of course). If a_i is the greatest separator element and a_{i+1} has a large prime factor then $[a_i, a_{i+1}] \geqq a_i\sqrt{N}$. Hence we again arrive at a contradiction by showing that there are at least $\sqrt{N}$ separators, or equivalently, there are at least $\sqrt{N}$ large primes which occur as factors of a_i-s.

We know that there are $(1+o(1))N\log 2$ numbers up to N having a large prime factor and we have $(1-\log 2+\varepsilon)N$ a_i-s [see (17)], hence at least εN a_i-s have a large prime factor. All of these a_i-s cannot be multiples of less than $\sqrt{N}$ large primes: indeed, the number of multiples up to N of $\sqrt{N}$ large primes is

$$\sum_{\text{„these } p\text{"}} \left[\frac{N}{p}\right] < \sum_{\sqrt{N} < p \leqq N^{1/2+\varepsilon/2}} \left[\frac{N}{p}\right] = \big(1+o(1)\big)N\log\frac{\frac{1}{2}+\frac{\varepsilon}{2}}{1/2} < \varepsilon N.$$

PROOF OF THEOREM 5. The permutation 1, 2, 6, 3, 12, 4, 20, 5, 35, 7, ... clearly satisfies (6), i.e., if we have already constructed a_{2n} and k is the smallest number which was not yet used, then a_{2n+1} should be a common multiple of a_{2n} and k (e.g. the smallest one still available) anf put $a_{2n+2}=k$.

To prove (7) we observe first the following facts:

Let $b_1, b_2, \ldots$ be arbitrary different natural numbers not greater than n. Then:

$$(b_1, b_2) \leqq \frac{n}{2}, \tag{18}$$

$$(b_1, b_2) \leqq \frac{n}{3} \quad \text{or} \quad (b_2, b_3) \leqq \frac{n}{3}, \tag{19}$$

$$\min_{1\leqq i\leqq 4} (b_i, b_{i+1}) \leqq \frac{n}{4}. \tag{20}$$

(18) is obvious. To show (19) assume indirectly that e.g. $\frac{n}{3} < d = (b_1, b_2) \geqq (b_2, b_3)$. Then either $b_1=d$ and $b_2=2d$ or $b_1=2d$ and $b_2=d$, but in both cases b_3 must be

at least $3d$ which is a contradiction. We can show (20) by similar methods. Put

$$d = \min_{l \leq i \leq 4} (b_i, b_{i+1}) = (b_k, b_{k+1}).$$

Arguing again indirectly we have $\{b_k, b_{k+1}\}$ $\{d, 2d, 3d\}$, and taking step by step the neighbouring numbers, and having in mind that the greatest common divisor must be at least d, we obtain that

$$\{b_1, \ldots, b_5\} \subseteq \left\{d, 2d, 3d, \frac{3}{2} d\right\}$$

which is a contradiction.

Now we are ready to prove (7). Assume that $(a_i, a_{i+1}) > \frac{1}{c} i$ if i is large enough. Then $\{a_1, a_2, \ldots, a_{cn}\} \supseteqq \{1, 2, \ldots, n\}$ if n is large enough. On the other hand, taking $a_{cn/2}, \ldots, a_{cn-1}, a_{cn}$, at mos tevery second number can be less than or equal to n, since

$$(a_i, a_{i+1}) > \frac{1}{c} i \geqq \frac{1}{c} \cdot \frac{cn}{2} = \frac{n}{2}$$

which is impossible by (18) if both a_i and a_{i+1} are less than or equal to n.

Similarly, using (19) we obtain that at most the two third part of $a_{cn/3}, \ldots, a_{cn/2}$ is not greater than n, and finally using (20) we conclude that at most the 4/5 part of $a_{cn/4}, \ldots, a_{cn/3}$ is smaller than n. Hence

$$n \leqq \left(cn - \frac{cn}{2}\right) \cdot \frac{1}{2} + \left(\frac{cn}{2} - \frac{cn}{3}\right) \cdot \frac{2}{3} + \left(\frac{cn}{3} - \frac{cn}{4}\right) \cdot \frac{4}{5} + \frac{cn}{4}$$

i.e. $\frac{1}{c} \leqq \frac{61}{90}$, as asserted.

REMARKS. 1. We can improve (7) somewhat, if we use further inequalities of the type (18), (19) and (20). But this does not seem to give a serious reduction, and also the discovery of the proper inequalities is not too easy. E.g. $\min_{l \leq i \leq 12} (b_i, b_{i+1}) \leqq \frac{n}{5}$, and here 12 cannot be replaced by 11, as shown by the numbers

$$81j,\ 162j,\ 108j,\ 54j,\ 216j,\ 72j,\ 144j,\ 48j,\ 96j,\ 192j,\ 128j,\ 64j$$

$$\left(n = 216j,\ d = 48j = \frac{2}{9} n\right).$$

2. We mention the following related problem, where we can determine the extremum exactly:

THEOREM 6.

$$\liminf_i \frac{\min\{a_i, |a_{i+1} - a_i|\}}{i} \leqq \frac{3}{4} \tag{21}$$

is true for any permutation, and we can construct a permutation where equality holds.

PROOF OF THEOREM 6. The following permutation shows the possibility of equality:

$$1,\ 2,\ 3,\ 6,\ 4,\ 8,\ 5,\ 10, \ldots,$$

i.e. we always take the smallest number still available followed by its double.

To prove (21) we assume indirectly that there is a permutation satisfying

(22)

$$a_i > \left(\frac{3}{4}+\varepsilon\right) i \quad \text{and} \quad |a_{i+1}-a_i| > \left(\frac{3}{4}+\varepsilon\right) i \quad \text{for} \quad i \geqq i_0 \quad \text{with a fix} \quad \varepsilon > 0.$$

Then all the numbers up to $\left(\frac{3}{4}+\varepsilon\right) N$ must occur among $a_1, \ldots, a_N$, if N is large enough. This also means that at least $\left(\frac{1}{4}+\varepsilon\right) N$ numbers smaller than $\left(\frac{3}{4}+\varepsilon\right) N$ must appear among $a_{N/2+1}, \ldots, a_N$. Thus we obtain an $i > \frac{N}{2}$, for which both a_i and a_{i+1} are smaller than $\left(\frac{3}{4}+\varepsilon\right) N$. Say $a_{i+1} > a_i$, then

$$\left(\frac{3}{4}+\varepsilon\right) N > a_{i+1} = a_i + (a_{i+1}-a_i) > 2\left(\frac{3}{4}+\varepsilon\right) i > 2\left(\frac{3}{4}+\varepsilon\right) \frac{N}{2}$$

which is a contradiction.

We note that the proof gives slightly more, since we did not make really use of the ε in (22).

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