SUBGRAPHS IN WHICH EACH PAIR OF EDGES LIES IN A SHORT COMMON CYCLE

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1. Introduction.

By a k-graph we mean a pair $G = (V,E)$ consisting of a finite set V of vertices and a collection E of distinct k-element subsets of V called the edges of G. Our object here is to show that each such k-graph with sufficiently many vertices and sufficiently many edges must contain a subgraph H (that is, a "sub-k-graph" H), also having many edges, and having the property that each pair of edges of H lies together in a common subgraph of G which is a type of k-graph "cycle". In particular for graphs (k=2) we show that each pair of edges of the subgraph H lies together in a cycle of length 4 or one of length 6 in H in the usual graph-theoretic sense, with any two edges of H which share a common vertex being in a cycle of length 4.

Our definition of a "k-cycle" for $k > 2$ involves the notion of a "separating edge" which was used by Lovász [6] in the formulation of his definition of a "k-forest".

2. The Main Results.

We shall use $G^k(n,\ell)$ to denote a k-graph having n vertices and ℓ edges, and $K^k(m,m,\cdots,m)$ to denote the complete k-partite k-graph having m vertices in each color class. Our first result, from which the rest will be derived, is that each k-graph with sufficiently many edges contains a large number of distinct subgraphs each of which is a $K^k(2,2,\cdots,2)$. The argument used is based on familiar techniques such as those employed in [3].

Theorem 1. For each positive constant c and sufficiently large n there exists a positive constant c' such that each $G^k(n,cn^k)$ contains $c'n^{2k}$

distinct copies of $K^k(2,2,\cdots,2)$.

Proof. We proceed by induction on k, considering first a graph $G = G^2(n,cn^2)$.

By standard results (cf. [1]) we have that G contains a bipartite graph B with vertices $\{x_1,x_2,\cdots,x_m\}$ and $\{y_1,y_2,\cdots,y_m\}$, $m = \lfloor n/2 \rfloor$, and c_1n^2 of the edges x_i,y_j, $1 \le i, j \le m$, for some positive constant c_1. Let $d(x_i)$ denote the degree of x_i in B and $d(y_j,y_\ell)$ the number of vertices x_i such that each of the edges x_iy_j and x_iy_ℓ are in B. Let $p = \sum_{\substack{j\neq\ell \\ 1\le j,\ell\le m}} d(y_j,y_\ell)$ and let q denote the number of copies of $K^2(2,2)$ in B. Then we have that $p = \sum_{i=1}^{m} \binom{d(x_i)}{2}$ and that $q = \sum_{\substack{j\neq\ell \\ 1\le j,\ell\le m}} \binom{d(y_j,y_\ell)}{2}$.

Since p is least if the $d(x_i)$ are all equal and q is least when the $d(y_i,y_\ell)$ are all the same, we have that $p \ge m\binom{2c_1n}{2}$ and that $q \ge \binom{m}{2}\binom{p/\binom{m}{2}}{2}$. Thus the result follows for k=2 when n is sufficiently large.

Now assume that the result is true for (k-1)-graphs $(k \ge 3)$ and consider a k-graph $G^k(n,cn^k)$. Again we may assume that our k-graph contains a k-partite sub-k-graph B having vertex set $V = \bigcup_{i=1}^{k} X^i$, $X^i = \{x_1^i,x_2^i,\cdots,x_m^i\}$, $m = \lfloor n/k \rfloor$, and c_1n^k of the edges $(x_{j_1}^1,x_{j_2}^2,\cdots,x_{j_k}^k)$ for some positive constant c_1. Let A_ℓ, $\ell = 1,2,\cdots,N = \binom{m}{2}^{k-1}$, denote those sets consisting of two vertices from each of the X^i, $2 \le i \le k$, and let $d(A_\ell)$ denote the number of vertices x_j^1 in X^1 for which $(x_j^1,x_{j_2}^2,x_{j_3}^3,\cdots,x_{j_k}^k)$ is an edge of B for each choice of $x_{j_2}^2,x_{j_3}^3,\cdots$, and $x_{j_k}^k$ from A_ℓ. Similarly let $\hat{d}(x_j^1)$ denote the number of the sets A for which $(x_j^1,x_{j_2}^2,x_{j_3}^3,\cdots,x_{j_k}^k)$ is an edge of B for each collection $x_{j_2}^2,x_{j_3}^3,\cdots,x_{j_k}^k$ chosen from A_ℓ. For some constant c_2 there are at least c_2m vertices in X^1 each contained in c_2n^{k-1} edges of B. For such a vertex $x^1_{j_1}$ the (k-1)-graph whose edges are all of the (k-1)-sets P_i for which $\{x^1_{j_1}\} \cup P_i$ is an edge of B has at least $c_2 n^{k-1}$ edges, and by the inductive hypothesis, contains at least $c_3 n^{2k-2}$ copies of $K^{k-1}(2,2,\cdots,2)$ for some positive constant c_3. For such a vertex we have $\hat{d}(x^1_{j_1}) \geq c_3 n^{2k-2}$. Since $\sum_{\ell=1}^{N} d(A_\ell) = \sum_{j=1}^{m} \hat{d}(x^1_j)$, it follows that $\sum_{\ell=1}^{N} d(A_\ell) \geq t$, where $t = c_4 n^{2k-1}$, for some positive constant c_4. Thus we have $\sum_{\ell=1}^{N} \binom{d(A_\ell)}{2} \geq N\binom{t/N}{2}$, from which the result follows for large n.

As a consequence of this theorem we have that for each positive constant c there exists a positive constant c' such that for sufficiently large n each $G^k(n,cn^k)$ contains an edge which is contained in at least $c'n^k$ distinct copies of $K^k(2,2,\cdots,2)$. (This fact for k = 2 could also be obtained as a nice application of the powerful graph-theoretic result of Szemerédi given in [8]). It is easily checked that a subgraph H of $G^2(n,cn^2)$ which consists of $c'n^2$ copies of $C_4 = K^2(2,2)$ all having a common edge xy has the property that each pair of edges of H are contained in a cycle of length 4 or one of length 6 in H. Any two edges of H which share a common vertex will be in a cycle of length 4, except possibly for some pairs where both edges contain the same vertex of the edge xy while neither contains both x and y. If we let H_1 be the subgraph consisting of those edges of H which do not meet xy, then for some positive constant c", H_1 contains $c''n^2$ distinct copies of C_4 all sharing a common edge x'y'. Let H_2 be the graph formed by adding to these $c''n^2$ copies of C_4 in H_1 the remaining edges of each C_4 in H which includes both the edge xy and some edge in H_1. It now follows that any two edges of H_2 which share a vertex lie in a cycle of length 4 in H_2 and so this subgraph has the properties described in the following result:

Corollary 1. For each positive constant c there exists a positive constant c' such that for sufficiently large n each $G^2(n,cn^2)$ contains a subgraph H with $c'n^2$ edges which has the property that each pair of edges of H are contained in a cycle of H of length 4 or 6 and each pair of edges which share a common vertex are in a cycle of length 4.

Before considering an analogue of this corollary for $k>2$ we must formulate an appropriate counterpart for the notion of a "cycle" in a graph. Our approach involves the following notion due to Lovász [6]. An edge E of a k-graph $G = (V,E)$ is a separating edge of G if there exists a partition of V into k classes such that E meets each class of this partition, but every other edge of G meets at most k-1 of these classes. (For k=2 a separating edge is simply a "cutedge" in the usual graph-theoretic sense). Lovász called a k-graph each of whose edges is a separating edge a k-forest and showed in [6] that a k-forest with n vertices has at most $\binom{n-1}{k-1}$ edges. In [9] Winkler showed that the (k-1)-dimensional simplices of a simplicial complex which triangulates a (k-1)-dimensional closed manifold, thought of as the edges of a k-graph, include no separating edges. Lovász obtained a more general result in [7] by considering a matroid of rank k defined on the vertices of such a simplicial complex. In [5] Lindström extended Lovász' theorem by allowing, in place of a (k-1)-manifold, a cycle of an arbitrary chain-complex, and also obtained new proofs of the earlier results of Lovász and Winkler. It follows from these results (or by a slight modification of the proof of Winkler's theorem in [9]) that a graph G which is such that each set of k-1 vertices is contained in an even number of edges has no separating edges. A k-graph G is called strongly connected provided that for each pair of edges E and F of G there exists a finite sequence of edges of G, $E = E_1, E_2, \cdots, E_\ell = F$ such that $|E_i \cap E_{i+1}| = k-1$ for $1 \le i \le \ell-1$. We shall use the term k-cycle to denote a k-graph with at least one edge, which has no separating edges and which is minimal with respect to this property. If an edge E of a k-graph G has a subset of k-1 vertices which is not a subset of any other edge of G, then it is easy to see that E is a separating edge of G. It follows that any strongly connected k-graph which is such that each set of k-1 vertices is contained in 0 or exactly 2 edges must be a k-cycle in our sense. (Note that these conditions applied to the highest dimensional simplices of a simplicial complex define a pseudomanifold in the sense of Brouwer and Lefschetz [4]. A k-cycle of this type would also be a circuit in the associated k-simplicial matroid over GF(2), but the relationship between our k-cycles in general and the matroid circuits is not clear). We may now formulate the following result:

Corollary 2. For each positive constant c there exists a positive constant c' such that for sufficiently large n each $G^k(n,cn^k)$ contains a sub-k-graph H with the property that each pair of edges of H are contained in a common k-cycle of H.

Proof. As for k = 2 the proof consists of showing that a subgraph consisting of $c'n^k$ copies of $K^k(2,2,\cdots,2)$ all sharing a common edge has the desired property. To see that this is so consider two edges E and F in such a subgraph and let the vertices of the k-edge common to all of the $K^k(2,2,\cdots,2)$ in this k-graph be $x_1,x_2,\cdots$, and x_k. If E and F are contained in the same copy of $K^k(2,2,\cdots,2)$, then this k-partite k-graph is the required k-cycle. Suppose then that E and F are in distinct $K^k(2,2,\cdots,2)$'s, say, Y and Z, and that the vertices of Y and Z, other than the x_i are $y_1,\cdots,y_k$ and $z_1,\cdots,z_k$, respectively, where for some r, $1 \leq r \leq k$, $y_i = z_i$ for $i = r+1,r+2,\cdots,k$ and $y_i \notin Z$, $z_i \notin Y$ for $1 \leq i \leq r$. Let X denote the k-graph obtained from $Y \cup Z$ by deleting all k-tuples containing $x_1,x_2,\cdots$, and x_r. Note that X contains both E and F. Let A denote a set of k-1 vertices which are contained in some edge of X. For exactly one value of j, $1 \leq j \leq k$, A does not contain x_j, y_j, or z_j. If A contains any y_i or z_i with $1 \leq i \leq r$, then A is contained in precisely two edges of X, one containing x_j and the other y_j or z_j (or $y_j = z_j$ if $j > r$). If A contains no y_i or z_i with $1 \leq i \leq r$, then we must have $j \leq r$, since otherwise A contains $x_1,x_2,\cdots$, and x_r. In this case A is again in two edges of X, one with y_j and the other with z_j. Thus in each case A is contained in exactly two edges of X.

To see that X is also strongly connected first note that each edge of X contains at least one y_i for $1 \leq i \leq r$ or at least one z_i, $1 \leq i \leq r$, but not both. An edge E containing some y_i, $1 \leq i \leq r$, is joined to the edge with vertices $y_1,\cdots,y_k$ by a sequence of edges where each successive edge is obtained from its predecessor by replacing one vertex by a vertex among $y_1,y_2,\cdots,y_k$. Similarly an edge containing a z_i, $1 \le i \le r$, is joined to the edge with vertices $z_1, z_2, \cdots$, and z_k. Now $(y_1, y_2, \cdots, y_k)$ and $(z_1, z_2, \cdots, z_k)$ can each be joined by a sequence of edges to an edge containing $x_2, x_3, \cdots$, and x_r. Finally, since the edges $(y_1, x_2, x_3, \cdots, x_r, y_{r+1}, \cdots, y_k)$ and $(z_1, x_2, x_3, \cdots, x_r, z_{r+1}, \cdots, z_k)$ share k-1 vertices, we have that X is strongly connected. By the remarks above, X must be a k-cycle which concludes the proof.

It was show by Brown, Erdös, and Sós in [2] that each 3-graph $G^3(n, cn^{5/2})$, for n sufficiently large, contains a simplicial complex which is a triangulated 2-sphere. An analysis of the proof of Corollary 2 shows that for k = 3 each pair of edges in the sub-k-graph of $G^2(n, cn^3)$ constructed are contained together in a triangulated 2-sphere in that subgraph.

Further Results and Problems.

It is not difficult to show that each $G^2(n, cnf(n))$ contains a subgraph with $c'(f(n))^2$ edges each two of which lie on some common cycle. The existence of graphs with large girth and fixed minimum degree (see [1], Chpt. 3) shows, however, that a $G^2(n, cnf(n))$ may contain no subgraph in which each pair of edges lie on a short common cycle. What conditions would insure the existence of a large subgraph in which each set of m edges, no three incident with the same vertex, all lie on a common cycle?

Many questions remain to be answered concerning k-forests and the graphs we have called k-cycles. In particular we have no characterization of these k-cycles. As indicated, each k-graph in which every set of k-1 vertices is contained in any even number of edges (and hence each circuit of a k-simplicial matroid over GF(2)) must contain a k-cycle. There exists a 3-cycle, however, (with 6 vertices and 13 edges) which contains no nonempty sub-3-graph in which each set of 2 vertices is contained in an even number of edges. (If each k-cycle did contain such a subgraph, the notions of "k-cycle" and "matroid circuit" would coincide).

A k-forest on n vertices has at most $\binom{n-1}{k-1}$ edges (cf. [5] or [6]).

There exist exactly two 3-forests on 5 vertices with 6 edges, one having as edges all 3-subsets of $\{1,2,3,4,5\}$ which contain 1 and the other having (123), (124), (125), (145), (234), and (235) as edges. The properties of those k-forests with the maximum possible number of edges have yet to be investigated. It is not known whether every collection of separating edges in a strongly connected k-graph with n vertices can be extended to a k-forest with $\binom{n-1}{k-1}$ edges, or whether, in a k-graph with many edges, the fraction of edges which are separating edges must be small.

It follows from Lindström's result [5] that the k-forests contained in a k-graph are independent sets in the k-simplicial matroid determined by H. Examples given in [5] show, however, that the collection of all k-forests in H need not be equal to the collection of independent sets for some matroid.

The 7-point projective plane, or any larger Steiner triple system, viewed as a 3-graph, shows that a 3-forest may not be 2-colorable.

References

[1] B. Bollobás, "Extremal Graph Theory," Academic Press, New York, 1978.

[2] W.G. Brown, P. Erdös, and V.T. Sós, On the Existence of Triangulated Spheres in 3-Graphs and Related Questions, Periodica Math. Hung., Vol. 3 (1973), 221-228.

[3] P. Erdös, On Extremal Problems of Graphs and Generalized graphs, Israel J. Math., Vol. 2 (1964), 183-190.

[4] S. Lefschetz, "Topology", Chelsea Pub. Co., New York, 1956.

[5] B. Lindström, On Matroids and Sperner's Lemma, Europ. J. Comb., Vol. 2 (1981), 65-66.

[6] L. Lovász, Topological and Algebraic Methods in Graph Theory in "Graph Theory and Related Topics" (Proc. Conf. Univ. Waterloo, Waterloo, Ont., 1977), pp. 1-14, Academic Press, New York, 1979.

[7] L. Lovász, Matroids and Sperner's Lemma, Europ. J. Comb., Vol. 1 (1980), 65-66.

[8] E. Szemerédi, Regular Partitions of Graphs, in "Proc. Colloq. Inter. C.R.N.S." (J.-C. Bermond, J.-C. Fournier, M. Das Vernas, D. Sotteau, Eds.), 1978, 399-401.

[9] P.M. Winkler, On Connectivity of Triangulations of Manifolds, Discrete Math., Vol. 32 (1980), 93-94.