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RAMSEY-MINIMAL GRAPHS FOR STAR-FORESTS

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It is shown that if G and H are star-forests with no single edge stars, then (G, H) is Ramsey-finite if and only if both G and H are single stars with an odd number of edges. Further $(S_m \cup kS_1, S_n \cup lS_1)$ is Ramsey-finite when m and n are odd, where S_i denotes a star with i edges. In general, for G and H star-forests, $(G \cup kS_1, H \cup lS_1)$ can be shown to be Ramsey-finite or Ramsey-infinite depending on the choice of G, H, k, and l with the general case unsettled. This disproves the conjecture given in [2] where it is suggested that the pair of graphs (L, M) is Ramsey-finite if and only if (1) either L or M is a matching, or (2) both L and M are star-forests of the type $S_m \cup kS_1$, m odd and $k \geq 0$.

1. Introduction

Let F, G and H be (simple) graphs. Write $F \rightarrow (G, H)$ to mean that if each edge of F is colored red or blue, then either the red subgraph of F, denoted $(F)_R$, contains a copy of G, or the blue subgraph, denoted $(F)_B$, contains a copy of H. The class of all graphs F (up to isomorphism) such that $F \rightarrow (G, H)$ has been studied extensively, e.g. the *generalized Ramsey number* $r(G, H)$ is the minimum number of vertices of a graph in this class.

A graph F will be called (G, H)*-minimal* if $F \rightarrow (G, H)$ but $F' \not\rightarrow (G, H)$ for each proper subgraph F' of F. If G, H and F have no isolated vertices, F' can be replaced by $F - e$, where e is any edge of F. Here $F - e$ denotes the graph with vertex set the same as F and edge set that of F less edge e. The class of (G, H)-minimal graphs will be denoted by $\mathcal{R}(G, H)$. The pair (G, H) will be called *Ramsey-finite* if $\mathcal{R}(G, H)$ is finite, and *Ramsey-infinite* otherwise.

Several recent papers discuss the problem of determining whether the pair (G, H) is Ramsey-finite (see [2, 3, 4, 7]). In particular Nešetřil and Rödl [7] showed that (G, H) is Ramsey-infinite if both G and H are 3-connected or if G and H are forests neither of which is a union of stars. It is shown in [4] that (G, H) is Ramsey-finite if G is a matching and H arbitrary. In addition, if (G, H) is Ramsey-finite for each graph H, then the results of [5] indicate that G must be a matching. The purpose of this paper is to discuss one of the remaining gaps, which is to determine whether (G, H) is Ramsey-finite or infinite whenever G and H are star-forests, i.e., a forest of stars.

At this point we introduce some further notation and terminology. The word "coloring" will always refer to coloring each edge of some graph red or blue. A coloring of F with neither a red G or blue H will be called *(G, H)-good*. The modifier (G, H) may be dropped when the meaning is clear. For notational convenience a (G, H)-good coloring of F will be frequently symbolized by $G \not\leq (F)_R$ and $H \not\leq (F)_B$. Here the symbol "$\leq$" is read "subgraph of". The degree of a vertex x in $(F)_R$ (or $(F)_B$) will be denoted by $d_R(x)$ (or $d_B(x)$). A cycle on n vertices $\{x_1, x_2, \ldots, x_n\}$ with x_i adjacent to x_{i+1} for each i will be denoted by $(x_1, x_2, \ldots, x_n, x_1)$. The symbol mG will refer to m disjoint copies of the graph G. *Also S_n will denote a star with n edges.* This notation, instead of the usual $K_{1,n}$, was selected because of its frequent appearance and its simplicity. Further notation will follow that of standard references [1] and [6].

2. Stars

In this section we decide whether (G, H) is Ramsey-finite or infinite in the special case in which G and H are stars. Since (G, H) is Ramsey-finite whenever G is a matching [4], we deal only with nontrivial stars, i.e., not single edge stars. We will show that (S_s, S_t) is Ramsey-infinite except when both s and t are odd, in which case $\mathcal{R}(S_s, S_t) = \{S_{s+t-1}\}$.

To begin we state a well-known "old" theorem which is used strongly in what follows.

Theorem 1 (Petersen [8]). *A connected graph G is 2-factorable if and only if it is regular of even degree.*

Theorem 2. *Let s and t be odd positive integers and let F be an arbitrary graph. If $\Delta(F) < s+t-1$, then F can be colored such that $S_s \not\leq (F)_R$ and $S_t \not\leq (F)_B$.*

Proof. Embed F in a regular graph F' of degree $s+t-2$. By Petersen's Theorem (Theorem 1) F' is 2-factorable when $s+t-2>0$, so color $(s-1)/2$ of the factors red and $(t-1)/2$ of the factors blue. Clearly $F' \not\rightarrow (S_s, S_t)$ so that $F \not\rightarrow (S_s, S_t)$.

Corollary 3. *If s and t are odd positive integers, then $\mathcal{R}(S_s, S_t) = \{S_{s+t-1}\}$.*

Proof. Clearly $S_{s+t-1} \in \mathcal{R}(S_s, S_t)$. Also if $F \in \mathcal{R}(S_s, S_t)$, then by Theorem 2, $\Delta(F) \geq s+t-1$. Hence $F \in \mathcal{R}(S_s, S_t)$ implies $S_{s+t-1} \leq F$, so that $F = S_{s+t-1}$.

Theorem 4. *If s and t are even positive integers, then (S_s, S_t) is Ramsey-infinite.*

Proof. Let l be an odd positive integer, $l \geq s+t-1$. Recall that K_l is the edge disjoint union of $(l-1)/2$ spanning cycles $G_1, G_2, \ldots, G_{(l-1)/2}$. Define F as the union of the cycles $G_1, G_2, \ldots, G_{(s+t-2)/2}$. Clearly F has l vertices and is regular of degree $s+t-2$. It is easy to see that $F \rightarrow (S_s, S_t)$. If this were not the case, then there would exist a coloring of F with $(F)_R$ regular of degree $s-1$ and $(F)_B$ regular of degree $t-1$. This is impossible since then both $(F)_R$ and $(F)_B$ have an odd number of vertices of odd degree. Furthermore if $e \in E(F)$, then $F-e \nrightarrow (S_s, S_t)$. To see this assume without loss of generality that $e \in E(G_{(s+t-2)/2})$. Then color alternating edges of the path $G_{(s+t-2)/2}-e$ together with all the edges of $G_1, G_2, \ldots, G_{(s-2)/2}$ red and the remaining edges of $F-e$ blue. This gives a good coloring of $F-e$. Hence we have shown that $F \in \mathcal{R}(S_s, S_t)$. Since l is any odd positive integer greater than $s+t-2$, the result follows.

Theorem 5. *Let s be odd* $(s \geq 3)$ *and t be an even positive integer. Then* (S_s, S_t) *is Ramsey-infinite.*

Proof. Let l be an odd positive integer, $l \geq s+t$. Then K_l is the edge disjoint union of $(l-1)/2$ spanning cycles $G_1, G_2, \ldots, G_{(l-1)/2}$. Suppose that G_1 is the cycle $(x_1, x_2, \ldots, x_l, x_1)$. Define the graph $F(\beta)$ as the edge disjoint union of the cycles $G_2, G_3, \ldots, G_{(s+t-1)/2}$ and the edges $\{x_2, x_3\}, \{x_4, x_5\}, \ldots, \{x_{l-1}, x_l\}$ of G_1, together with free edge β attached at vertex x_1, i.e., edge β has one of its end vertices identified with x_1 and the other end vertex remains of degree 1 in $F(\beta)$. Thus $F(\beta)$ is a graph on $l+1$ vertices, l of them of degree $s+t-2$, and the remaining vertex (an end vertex of β) is of degree 1.

We show that $F(\beta)$ can be colored such that $S_s \nsubseteq (F(\beta))_R$ and $S_t \nsubseteq (F(\beta))_B$, but under such colorings β is colored blue. To see that such a coloring exists, color the edges of $G_2, G_3, \ldots, G_{(s+1)/2}$ red and the remaining edges blue. Note that under this coloring β is colored blue. Also under all good colorings of $F(\beta)$ each of the l vertices of degree $s+t-2$ must be of red degree $s-1$ and blue degree $t-1$. Thus edge β is colored blue, otherwise $(F(\beta)-\beta)_B$ is a graph on l vertices, regular of degree $t-1$, i.e., has an odd number of vertices of odd degree. We have shown that $F(\beta)$ has good colorings, but under all such colorings β is colored blue.

Next we show $F(\beta)$ is minimal with respect to the property that under good colorings β is colored blue. By this we mean that if $e \in E(F(\beta))$, $e \neq \beta$, then $F(\beta)-e$ has a good coloring with β colored red. To establish this let $e \in E(F(\beta))$, $e \neq \beta$. Since $s \geq 3$, let G_2 be the cycle $(y_1, y_2, \ldots, y_l, y_1)$. Without loss of generality assume $e \in E(G_1 \cup G_2)$ and that e is incident to y_1. Then color the edges $\{y_2, y_3\}$, $\{y_4, y_5\}, \ldots, \{y_{l-1}, y_l\}$ of G_2 and all the edges of $G_{(s+3)/2}, G_{(s+5)/2}, \ldots, G_{(s+t-1)/2}$ blue. This remaining edges of $F(\beta)-e$ are colored red. This coloring is a (S_s, S_t)-good coloring of $F(\beta)-e$ with edge β colored red.

We now take t copies of $F(\beta)$, call them $F(\beta_1), F(\beta_2), \ldots, F(\beta_t)$, and identify the vertices of degree one. Call this graph G and name the identified vertex v, i.e., G has the vertex v with incident edges $\beta_1, \beta_2, \ldots, \beta_t$.

Observe that $G \to (S_s, S_t)$, since the only good colorings of the $F(\beta_i)$ would make all β_i blue giving a blue S_t with central vertex v. Also for $e \in E(G)$, $G-e$ can be given a (S_s, S_t)-good coloring. If $e \in F(\beta_1)$ give $F(\beta_1)-e$ the good coloring described above with β_1 (if present) colored red and $F(\beta_i)$, $i \geq 2$, the good coloring described above with β_i colored blue. This coloring shows $G-e$ can be good colored so that $G-e \not\to (S_s, S_t)$. Hence $G \in \mathcal{R}(S_s, S_t)$.

Since l is any odd positive integer, $l \geq s+t$, we have that $R(S_s, S_t)$ is infinite.

3. Star-forests

In this section we consider the more general pair

$$\left(\bigcup_{i=1}^{s} S_{n_i}, \bigcup_{j=1}^{t} S_{m_j}\right), \quad s \geq 2 \text{ or } t \geq 2,$$

and ask whether it is Ramsey-infinite. This is answered affirmatively when all the stars are nontrivial, i.e., not single edges. In light of the results of the previous section and the previously mentioned result that (mS_1, H) is Ramsey-finite for arbitrary H, one might expect, if M and L are matchings, that $(G \cup M, H \cup L)$ is Ramsey-finite if and only if (G, H) is Ramsey-finite. We shall see this isn't the case even when G and H are star-forests.

Lemma 6. *Let $F_1 = \bigcup_{i=1}^{s} S_{n_i}$ and $F_2 = \bigcup_{j=1}^{t} S_{m_j}$ with $n_1 \geq n_2 \geq \cdots \geq n_s$ and $m_1 \geq m_2 \geq \cdots \geq m_t$. Let $g_l = \max\{n_i + m_j - 1 \mid i+j = l+1\}$ for $l = 1, 2, \ldots, k$, $k \leq s+t-1$. Then*

$$\left(\bigcup_{l=1}^{k} S_{g_l}\right) \to \left(\bigcup_{i=1}^{z} S_{n_i}, \bigcup_{j=1}^{k-z+1} S_{m_j}\right) \quad \text{for } z \leq s \text{ and } 1 \leq k-z+1 \leq t.$$

In particular if $z = s$ and $k = s+t-1$, then

$$\left(\bigcup_{l=1}^{s+t-1} S_{g_l}\right) \to (F_1, F_2).$$

Proof. Color $\bigcup_{l=1}^{k} S_{g_l}$. Assume for some r, $r < z$, that $\bigcup_{i=1}^{r} S_{n_i} < (\bigcup_{l=1}^{k} S_{g_l})_R$ but $\bigcup_{i=1}^{r+1} S_{n_i} \not\leq (\bigcup_{l=1}^{k} S_{g_l})_R$. Since the g_l are nonincreasing, we can assume without loss of generality that $S_{n_i} \leq (S_{g_i})_R$ for $1 \leq i \leq r$. Therefore $S_{n_{r+1}} \not\leq (\bigcup_{l=r+1}^{k} S_{g_l})_R$. But $g_l \geq n_{r+1} + m_{l-r} - 1$ for $l = r+1, r+2, \ldots, r+k-z+1$. Hence $S_{m_{l-r}} \leq (S_{g_l})_B$ for $l = r+1, r+2, \ldots, r+k-z+1$, so that $\bigcup_{i=1}^{z} S_{n_i} \not\leq (\bigcup_{l=1}^{k} S_{g_l})_R$ implies that

$$\bigcup_{j=1}^{k-z+1} S_{m_j} \leq \left(\bigcup_{l=1}^{k} S_{g_l}\right)_B.$$

Lemma 7. *The pair $(S_s \cup S_t, S_l)$ is Ramsey-infinite for $s, t, l \geq 2$.*

Proof. We assume throughout the proof that $s \geq t$. Consider a disjoint family of sets $\{A_i\}_{i=1}^k$ (k even, $k \geq 6$) with

$$|A_1| = s+t-1, \quad |A_2| = t, \quad |A_i| = t(l-1) \quad \text{for } i = 3, \ldots, k-2,$$
$$|A_{k-1}| = t, \quad |A_k| = 1.$$

Let $G = G(s, t, l, k)$ be the graph with vertex set $\bigcup_{i=1}^k A_i$, each A_i an independent set in G, such that each of the following hold:

(1) The pairs (A_1, A_2) and (A_{k-1}, A_k) generate complete bipartite graphs.

(2) The pair (A_i, A_{i+1}) generates a regular bipartite graph of degree $t+l-3$ when i is odd ($3 \leq i \leq k-3$) and regular of degree 1 when i is even ($4 \leq i \leq k-4$).

(3) The pairs (A_2, A_3) and (A_{k-2}, A_{k-1}) generate bipartite graphs with the vertices of $A_2(A_{k-1})$ of degree $l-1$ and the vertices of $A_3(A_{k-2})$ of degree 1. (This degree is relative to the subgraphs generated by the pairs (A_2, A_3) and (A_{k-2}, A_{k-1}).)

The graph G has no edges other than those indicated in (1), (2) and (3) above and is shown for $s = 5$, $l = 3$, $t = 3$, and $k = 8$ in Fig. 1.

Color G and suppose that G contains no red $S_s \cup S_t$ and no blue S_l. First note that $d(x) = s+t+l-2$ for $x \in A_2$. Since $S_l \not\leq (G)_B$, $d_R(x) \geq s+t-1$ for $x \in A_2$. Also $S_s \cup S_t \not\leq (G)_R$ so that the number of vertices collectively adjacent in $(G)_R$ to any two distinct vertices in A_2 is at most $s+t-1$. Hence all the edges between vertices of A_1 and A_2 are red and between A_2 and A_3 are blue. This implies that the pair (A_3, A_4) generates a regular bipartite graph of degree $t-1$ in $(G)_R$ and a regular bipartite graph of degree $l-2$ in $(G)_B$. Then all the edges between vertices of A_4 and A_5 are blue. Hence the coloring of the edges between all pairs (A_i, A_{i+1}) are determined for $i \leq k-3$. They are colored like those between the pair (A_3, A_4) if i is odd and like those between the pair (A_4, A_5) when i is even. This implies that the edges between A_{k-2} and A_{k-1} are blue, which in turn forces the edges between A_{k-1} and the vertex of A_k to be colored red. This gives $S_s \cup S_t \leq (G)_R$, a contradiction. Hence $G \to (S_s \cup S_t, S_l)$.

Next let $e = \{x_i, x_{i+1}\} \in E(G)$, $x_i \in A_i$, $x_{i+1} \in A_{i+1}$, $i \geq 2$. Consider the case when e is colored red in the coloring given above. Under this coloring there exists a

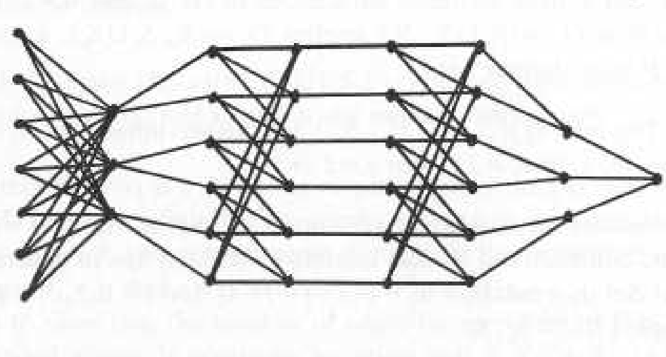

Fig. 1.

path with vertices $x_i, x_{i+1}, \ldots, x_k$, where $x_j \in A_j$ for each j, with the edges $\{x_i, x_{i+1}\}$, $\{x_{i+2}, x_{i+3}\}, \ldots, \{x_{k-1}, x_k\}$ in $E((G)_R)$ and the edges $\{x_{i+1}, x_{i+2}\}$, $\{x_{i+3}, x_{i+4}\}, \ldots, \{x_{k-2}, x_{k-1}\}$ in $E((G)_B)$. Replace this red-blue alternately colored path by a blue-red alternately colored one, i.e., interchange the colors on this path leaving unchanged the rest of G as colored. The case when e is blue is handled similarly. It follows that $G-e$ under this modified coloring is $(S_s \cup S_t, S_l)$-good. Thus $G-e \not\rightarrow (S_s \cup S_t, S_l)$. Thus removing appropriate edges between A_1 and A_2 gives a graph $G' \in \mathcal{R}(S_s \cup S_t, S_l)$ of diameter $k-1$. Since k can be taken arbitrarily large we have that $\mathcal{R}(S_s \cup S_t, S_l)$ is an infinite set.

Lemma 8. *Let u, w, r, z be positive integers with $u \geq w \geq 2$, $r \geq z \geq 2$. Set*

$$A = \{F \in \mathcal{R}(S_u \cup S_w, S_z) \mid F \rightarrow (S_w, S_r \cup S_z)\},$$

$$B = \{F \in \mathcal{R}(S_w, S_r \cup S_z) \mid F \rightarrow (S_u \cup S_w, S_z)\}.$$

Then either A or B has infinitely many elements.

Proof. Without loss of generality assume $z \geq w$. Suppose neither A or B have infinitely many elements, and let k be chosen so that $k-1$ exceeds the diameter of all the graphs in $A \cup B$. Let $G_1 = G(u, w, z, k)$ and $G_2 = G(r, z, w, k)$ where $G(s, t, l, k)$ is the graph G defined in the proof of Lemma 7. Since $G_2 \rightarrow (S_w, S_r \cup S_z)$ and all subgraphs of G_2 in $\mathcal{R}(S_w, S_r \cup S_z)$ are of diameter $k-1$ we have that $G_2 \not\rightarrow (S_u \cup S_w, S_z)$, otherwise G_2 contains a subgraph of diameter $k-1$ in $A \cup B$. Take a $(S_u \cup S_w, S_z)$-good coloring of G_2 and select distinct vertices $x, y \in A_2$ of the graph G_2. Since $d(x) = d(y) = r+z+w-2$ and $S_z \not\subseteq (G_2)_B$, $d_R(x)$ and $d_R(y)$ are both at least $r+w-1$. But $S_u \cup S_w \not\subseteq (G_2)_R$ so that $r+w-1 \leq u+w-1$, giving that $u \geq r$. Also $G_1 \rightarrow (S_u \cup S_w, S_z)$, and all subgraphs of G_1 in $\mathcal{R}(S_u \cup S_w, S_z)$ are of diameter $k-1$, so that as above $G_1 \not\rightarrow (S_w, S_r \cup S_z)$. Give G_1 a $(S_w, S_r \cup S_z)$-good coloring and select distinct vertices $x, y \in A$ of the graph G_1. Since $d(x) = d(y) = u+w+z-2$ and $S_w \not\subseteq (G_1)_R$, $d_B(x)$ and $d_B(y)$ are both at least $u+z-1 \geq r+z-1$. But $S_r \cup S_z \not\subseteq (G_1)_B$ so that $d_B(x) = d_B(y) = r+z-1$, which means that x and y have common adjacencies in $(G_1)_B$ and $u=r$. This implies that $w=z$ so that $G_1 \rightarrow (S_u \cup S_w, S_z)$ implies $G_1 \rightarrow (S_w, S_r \cup S_z)$, a contradiction. Hence A or B is an infinite set.

Theorem 9. *The pair $(\bigcup_{i=1}^{s} S_{n_i}, \bigcup_{j=1}^{t} S_{m_j})$ is Ramsey-infinite for $n_1 \geq n_2 \geq \cdots \geq n_s \geq 2$, $m_1 \geq m_2 \geq \cdots \geq m_t \geq 2$, when $s \geq 2$ or $t \geq 2$.*

Proof. First consider the case when $s \geq 2$ and $t \geq 2$. Set $u = n_{s-1}$, $w = n_s$, $r = m_{t-1}$, and $z = m_t$ and define A and B as in Lemma 8. Without loss of generality assume A is infinite. Set $g_l = \max\{n_i + m_j - 1 \mid i+j = l+1\}$ for $l = 1, 2, \ldots, s+t-3$ and color the graph $\bigcup_{l=1}^{s+t-3} S_{g_l}$. If

$$\bigcup_{i=1}^{s} S_{n_i} \not\subseteq \left(\bigcup_{l=1}^{s+t-3} S_{g_l}\right)_R \quad \text{and} \quad \bigcup_{j=1}^{t} S_{m_j} \not\subseteq \left(\bigcup_{l=1}^{s+t-3} S_{g_l}\right)_B,$$

then by Lemma 6 we have

$$\bigcup_{i=1}^{s-1} S_{n_i} \leqslant \Big(\bigcup_{l=1}^{s+t-3} S_{g_l}\Big)_R \quad \text{and} \quad \bigcup_{j=1}^{t-2} S_{m_j} \leqslant \Big(\bigcup_{l=1}^{s+t-3} S_{g_l}\Big)_B,$$

or

$$\bigcup_{i=1}^{s-2} S_{n_i} \leqslant \Big(\bigcup_{l=1}^{s+t-3} S_{g_l}\Big)_R \quad \text{and} \quad \bigcup_{j=1}^{t-1} S_{m_j} \leqslant \Big(\bigcup_{l=1}^{s+t-3} S_{g_l}\Big)_B.$$

Without loss of generality assume the former occurs. Take $H \in A$ and color it. Since $S_{n_s} \leqslant (H)_R$ or $S_{m_{t-1}} \cup S_{m_t} \leqslant (H)_B$ it follows that

$$\Big(\bigcup_{l=1}^{s+t-3} S_{g_l}\Big) \cup H \to \Big(\bigcup_{i=1}^{s} S_{n_i}, \bigcup_{j=1}^{t} S_{m_j}\Big).$$

Next let $e \in E(H)$ and give $H - e$ a $(S_{n_{s-1}} \cup S_{n_s}, S_{m_t})$-good coloring. Color the $\bigcup_{l=1}^{s-2} S_{g_l}$ red and color the $\bigcup_{l=s-1}^{s+t-3} S_{g_l}$ blue. Clearly this coloring gives a $(\bigcup_{i=1}^{s} S_{n_i}, \bigcup_{j=1}^{t} S_{m_j})$-good coloring of $(\bigcup_{l=1}^{s+t-3} S_{g_l}) \cup (H - e)$. Since A is infinite we deduce that $\mathcal{R}(\bigcup_{i=1}^{s} S_{n_i}, \bigcup_{j=1}^{t} S_{m_j})$ is infinite when both $s \geqslant 2$ and $t \geqslant 2$.

The proof when $s = 1$ or $t = 1$ is similar. Without loss of generality assume $t = 1$ so that $s \geqslant 2$. Let $H \in \mathcal{R}(S_{n_{s-1}} \cup S_{n_s}, S_{m_1})$. Observe as in the first case

$$\Big(\bigcup_{l=1}^{s-2} S_{g_l}\Big) \cup H \to \Big(\bigcup_{i=1}^{s} S_{n_i}, S_{m_1}\Big) \quad \text{and} \quad \Big(\bigcup_{l=1}^{s-2} S_{g_l}\Big) \cup (H - e) \nrightarrow \Big(\bigcup_{i=1}^{s} S_{n_i}, S_{m_1}\Big),$$

where $e \in E(H)$ and $g_l = n_l + m_1 - 1$. Since $(S_{n_{s-1}} \cup S_{n_s}, S_{m_1})$ is Ramsey-infinite by Lemma 7, we have that $(\bigcup_{i=1}^{s} S_{n_i}, S_{m_1})$ is Ramsey-infinite also. This completes the proof of the theorem.

We next investigate whether (G, H) is Ramsey-finite or Ramsey-infinite when G and H are star-forests with some of the stars trivial (single edges). Unfortunately our results are incomplete and indicate that the complete solution of the problem could be difficult.

Theorem 10. *The pair* $(S_{s_1} \cup t_1 S_1, S_{s_2} \cup t_2 S_1)$ *is Ramsey-finite when both* s_1 *and* s_2 *are odd positive integers, and* t_1 *and* t_2 *are nonnegative integers.*

Proof. If either s_1 or s_2 is 1, then the result follows from [4], where it is proved that (mS_1, H) is Ramsey-finite for all graphs H. Also if $t_1 = t_2 = 0$, then the result is that of Corollary 3. Hence we assume throughout the proof that $s_1 \geqslant s_2 \geqslant 3$ and setting $t = \max\{t_1, t_2\}$, that $t \geqslant 1$. We also let $t^* = \max\{t_1 + t_2, t_1 + 1, t_2 + 1\}$.

It suffices to show that the number of edges for members of $\mathcal{R}(S_{s_1} \cup t_1 S_1, S_{s_2} \cup t_2 S_1)$ is bounded above. In particular we show that if $F \in \mathcal{R}(S_{s_1} \cup t_1 S_1, S_{s_2} \cup t_2 S_1)$ then $|E(F)| \leqslant k^2 t^* + 1$ where $k = 4t + 2s_1 - 1$. We remark that this upper bound is undoubtedly not the best possible, only a convenient one.

The proof is by contradiction, so suppose there exists an $F \in \mathcal{R}(S_{s_1} \cup t_1 S_1, S_{s_2} \cup t_2 S_1)$ such that $|E(F)| > k^2 t^* + 1$. Let v be a vertex with $d(v) = \Delta(F)$. Since s_1 and s_2 are both odd, Theorem 2 implies that $d(v) \geq s_1 + s_2 - 1$.

Assume for the moment that $d(v) > k$. Remove an edge e incident to v and give $F - e$ a good coloring. Then $d_R(v) \geq 2t + s_1$ or $d_B(v) \geq 2t + s_1$, so assume the former. If e is colored red and $F - e$ keeps its good coloring, then $S_{s_1} \cup t_1 S_1 \leq (F)_R$. Thus in $(F-e)_R$ either $t_1 S_1$ or $S_{s_1} \cup (t_1 - 1) S_1$ is disjoint from v. But $t_1 S_1$ is incident to at most $2t_1$ neighbors of v in $(F-e)_R$ and $S_{s_1} \cup (t_1 - 1) S_1$ is incident to at most $s_1 + 2t - 1$. Thus $d_R(v) \geq 2t + s_1$ in $F - e$ implies, in either case, that $S_{s_1} \cup t_1 S_1 \leq (F-e)_R$, a contradiction. Hence $d(v) = \Delta(F) \leq k$.

We next show that each edge of F is incident to a vertex of degree s_2 or more. Suppose this were not the case. Let e be an edge incident to vertices of degree less than s_2, and consider a good coloring of $F - e$. It must happen that $S_{s_1} \cup (t_1 - 1) S_1 \leq (F-e)_R$ and $S_{s_2} \cup (t_2 - 1) S_1 \leq (F-e)_B$. This implies that each edge in $(F-e)_R$ is incident to or part of any collection of t_1 disjoint stars in $(F-e)_R$ and each edge in $(F-e)_B$ is incident to or part of any collection of t_2 disjoint stars in $(F-e)_B$. Since $\Delta(F) = k$, the number of edges in a star together with edges incident to the star is at most k^2. Thus there are at most $k^2 t_1$ edges in $(F-e)_R$ and at most $k^2 t_2$ edges in $(F-e)_B$ implying that $|E(F-e)| \leq k^2(t_1 + t_2)$. This contradicts $|E(F)| > k^2 t^* + 1$, so that each edge of F is incident to a vertex of degree s_2 or more.

Next we show that there exists an edge of F whose end vertices are both of degree less than s_1. Suppose this were not the case. Then by removing an edge e with end vertices different from v, $F - e$ would contain at least $t^* + 1$ disjoint stars, t^* of them of degree s_1 or more, since as in the previous discussion t^* disjoint stars can account for at most $k^2 t^*$ edges. But $d(v) \geq s_1 + s_2 - 1$ in $F - e$ and $F - e$ contains at least $t^* + 1$ disjoint stars, t^* of them of degree s_1 or more, so that $F - e \to (S_{s_1} \cup t_1 S_1, S_{s_2} \cup t_2 S_1)$, a contradiction. Hence there exists an edge $f \in E(F)$ whose end vertices are of degree less than s_1.

Give $F - f$ a good coloring. Then $S_{s_1} \cup (t_1 - 1) S_1 \leq (F-f)_R$. But each edge of F is incident to a vertex of degree s_2 or more and $|E(F-f)| \geq k^2 t^* + 1$ so that $F - e$ has at least $t^* + 1$ disjoint stars with at least t^* of them of degree s_2 or more. This together with $S_{s_1} \leq (F-f)_R$ implies that the coloring given $F - f$ is not good, a contradiction. Hence the original supposition $|E(F)| > k^2 t^* + 1$ is false and the proof is complete.

Theorem 11. *Let l, n and s be positive integers with l and n odd and $n \geq l + s - 1$. Then the pair $(S_n \cup S_s, S_l \cup kS_1)$ is Ramsey-finite for $k \geq (n + 2l + s - 2)^2 + 1$.*

Proof. As in the proof of Theorem 10 it sufficies to show that members of $\mathcal{R}(S_n \cup S_s, S_l \cup kS_1)$ have a bounded number of edges. We show that if $F \in \mathcal{R}(S_n \cup S_s, S_l \cup kS_1)$, then

$$|E(F)| \leq (k+1)(c^3 + c) + (n-1)^2(k + 2c)$$

where $c = n + 2k + l + s$. Since $\mathcal{R}(H, mS_1)$ is finite, we assume throughout the proof that $l > 1$.

Suppose there exists an

$$F \in \mathcal{R}(S_n \cup S_s, S_l \cup kS_1)$$

with $|E(F)| > (k+1)(c^3 + c) + (n-1)^2(k+2c)$. By Theorem 2 we have $\Delta(F) \geq n + l - 1$.

Next we show by an argument similar to the one given in Theorem 10 that $\Delta(F) \leq c$. To see this let $v \in V(F)$ such that $d(v) = \Delta(F)$ and suppose $d(v) \geq c + 1$. Remove an edge e incident to v and give $F - e$ a good coloring. Then $d_R(v) \geq n + s + 1$ or $d_B(v) \geq 2k + l$ in $F - e$. If $d_R(v) \geq n + s + 1$, then color e red with $F - e$ keeping its good coloring. Since $S_n \cup S_s \leq (F)_R$, this means that either S_n or S_s is a subgraph of $(F)_R$ disjoint from v. But S_n and S_s contain $n + 1$ and $s + 1$ vertices respectively, so that $d_R(v) \geq n + s + 1$ in $F - e$ insures $S_n \cup S_s \leq (F - e)_R$ with v as central vertex of one of the stars. This contradicts the assumption that the coloring of $F - e$ is good. Likewise if $d_B(v) \geq 2k + l$ in $F - e$, it follows that $S_l \cup kS_1 \leq (F - e)_B$, a contradiction. Hence $\Delta(F) \leq c$.

Let $e = \{u, v\} \in E(F)$. If $d(u) < s$ and $d(v) < s$ then a good coloring for $F - e$ can be extended to a good coloring for F by coloring edge e red. Hence each edge of F is incident to a vertex of degree s or more.

We next calculate bounds on the number of vertices of F of degree n or more. For convenience let w denote this number. Clearly $w \geq k + 1$, for otherwise color all edges incident to anyone of these w vertices blue and all other edges of F red, yielding a good coloring of F.

To calculate an upper bound on w, let t be maximal such that $S_{n+l-1} \cup tS_n \leq F$. Note that $t \leq k$, since $n > s$ and

$$S_{n+l-1} \cup kS_n \cup S_s \in \mathcal{R}(S_n \cup S_s, S_l \cup kS_1).$$

Each vertex of degree n or more must have an incident edge which is also incident to a vertex of $S_{n+l-1} \cup tS_n$. Since $\Delta(F) \leq c$, there are at most $(t+1)(c^2+1)$ such vertices. Hence $k + 1 \leq w \leq (k+1)(c^2+1)$.

Let $H = \langle\{e \in E(F) \mid e = \{x, y\}$ and $\max\{d(x), d(y)\} \geq n\}\rangle$ and $T = \{v \in H \mid d(v) \geq n\}$. Since $|T| = w \leq (k+1)(c^2+1)$ and $\Delta(F) \leq c$ the number of edges assumed in F implies that there exists an $e \in E(F) - E(H)$. Give $F - e$ a good coloring and observe that $S_n \leq (F - e)_R \cap H$. We wish to show that $S_l \leq (F - e)_B \cap H$. Select $v \in T$ such that $d_R(v) = \Delta((F - e)_R)$. If $d(v) \geq n + l + s$, then since $w \geq k + 1$, $n \geq l + s - 1$, and $S_n \cup S_s \not\leq (F - e)_R$, we have $S_l \leq (F - e)_B \cap H$. If $d(v) \leq n + l + s - 1$, then $d_R(z) \leq n + l + s - 1$ for each $z \in T$. But $w \geq k + 1$ and $k \geq (n + 2l + s - 2)^2 + 1$ implies the existence of a vertex $u \in T$ such that $d(u) \geq n + 2l + s - 1$ or the existence of two disjoint stars in H, one of which is a red S_n. In either case we have $S_l \leq (F - e)_B \cap H$. Thus under the good coloring of $F - e$, we have $S_n \leq (F - e)_R \cap H$ and $S_l(F - e)_B \cap H$ with the centers of these stars in T.

Finally since $|E(F)| > (k+1)(c^3 + c) + (n-1)^2(k + 2c)$, $|T| \leq (k+1)(c^2+1)$, and

$\Delta(F) \leq c$, there are at least $(n-1)^2(k+2c)$ edges of $F-e$ which are outside of H. But $d(z) \leq n-1$ for $z \in V(F)-T$ and each edge of F is incident to a vertex of degree s or more. Hence there exist at least $k+2c$ disjoint stars of degree s or more outside of T. Since $\Delta(F) \leq c$, at least k of these disjoint stars are themselves disjoint from the S_n in $(F-e)_R$ and the S_l in $(F-e)_B$ exhibited in the last paragraph. Since all of these stars are in $F-e$, it follows that $S_n \cup S_s \leq (F-e)_R$ or $S_l \cup kS_1 \leq (F-e)_B$, a contradiction. This final contradiction completes the proof of the theorem.

Theorem 12. *Let l, n and s be positive integers with l and n odd, $n \geq s \geq 2$, $l \geq 2$, and $n < l+s-1$. Then the pair $(S_n \cup S_s, S_l \cup kS_1)$ is Ramsey-infinite for all non-negative integers k.*

Proof. Let t be an even integer, $t \geq 6$, and let $G = G(n, s, l, t)$ where G is the graph constructed in the proof of Lemma 7. It is easy to see that each subgraph G' of G, $G' \in \mathcal{R}(S_n \cup S_s, S_l)$, has diameter $t-1$ and besides $G' \to (S_n, S_l \cup S_1)$. Set $k^* = \max\{0, k-1\}$. Then since $G' \to (S_n \cup S_s, S_l)$ and $G' \to (S_n, S_l \cup S_1)$ it follows that $G' \cup k^*S_n \cup S_s \to (S_n \cup S_s, S_l \cup kS_1)$. Also for $e \in E(G')$ give $G'-e$ a $(S_n \cup S_s, S_l)$-good coloring and color $l-1$ edges of each star in the $k^*S_n \cup S_s$ blue and the remaining edges red. This clearly gives a $(S_n \cup S_s, S_l \cup kS_1)$-good coloring of $(G'-e) \cup k^*S_n \cup S_s$. Thus, since t is any even integer ($t \geq 6$) it follows that $(S_n \cup S_s, S_l \cup k_{S1})$ is Ramsey-infinite, completing the proof.

Let $\{H_i\}_{i=1}^m$ and $\{G_i\}_{i=1}^n$ be families of connected graphs with $(H_{i'}, G_{j'})$ Ramsey-infinite for some i' and j'. It seems reasonable to expect $(\bigcup_{i=1}^m H_i, \bigcup_{i=1}^n G_i)$ to be Ramsey-infinite. Theorem 11 together with Theorem 5 shows that this is not the case. In particular, in Theorem 11 let s be even and l odd ($l \geq 3$). Then by Theorem 5, (S_s, S_l) is Ramsey-infinite but $(S_n \cup S_s, S_l \cup kS_1)$ is Ramsey-finite for $k \geq (n+2l+s-2)^2+1$. This example is yet another indication that it is difficult to determine whether a pair of graphs is Ramsey-finite or Ramsey-infinite.

Our results are complete when G and H are star-forests with no single edge stars. In fact we have shown for such G and H that (G, H) is Ramsey-finite if and only if both G and H are single stars with an odd number of edges (Theorems 4, 5, 9 and Corollary 3). Further we have shown that when G and H are star-forests with no single-edge stars and with (G, H) Ramsey-finite, then $(G \cup kS_1, H \cup tS_1)$ is also Ramsey-finite (Theorem 10). We have failed to determine whether or not $(G \cup kS_1, H \cup tS_1)$ is Ramsey-finite or infinite for arbitrary star-forests G and H, although it can be shown to be Ramsey-infinite for large classes of star-forests. The special case when the pair is $(S_n \cup S_s, S_l \cup kS_1)$, $n \geq s$, n and l odd, k large, is completely settled in Theorems 11 and 12. In particular, since $(S_n \cup S_s, S_l)$ is Ramsey-infinite for $n \geq s \geq 2$ and $l \geq 2$, it would be of interest to find the largest integer k_0 such that $(S_n \cup S_s, S_l \cup k_0 S_1)$ is Ramsey-finite, n and l odd, $n \geq l+s-1$ (see Theorem 11). This leaves the following questions. *For what star-forests G and*

H and what positive integers k and t is $(G \cup kS_1, H \cup tS_1)$ *Ramsey-finite? In particular, if* (G, H) *is Ramsey-finite, is* $(G \cup kS_1, H \cup tS_1)$ *Ramsey-finite?*

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