RESIDUALLY-COMPLETE GRAPHS

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If G is a graph such that the deletion from G of the points in each closed neighborhood results in the complete graph K_n, then we say that G is K_n-residual. Similarly, if the removal of m consecutive closed neighborhoods yields K_n, then G is called m-K_n-residual. We determine the minimum order of the m-K_n-residual graphs for all m and n. The minimum order of the connected K_n-residual graphs is found and all the extremal graphs are specified.

1. Introduction

A graph G is said to be F-residual if for every point u in G, the graph obtained by removing the closed neighborhood of u from G is isomorphic to F. We inductively define multiply-F-residual graphs by saying that G is *m-F-residual* if the removal of the closed neighborhood of any point of G results in an $(m-1)$-F-residual graph, where of course a 1-F-residual graph is simply an F-residual graph.

We are concerned with residually-complete graphs, i.e., graphs which are m-K_n-residual for some m and n. It is easy to see that there exists such a graph for any m and n, since $(m+1)K_n$ is clearly such a graph. Actually we show that there exist infinitely many connected m-K_n-residual graphs for any m and n.

It is natural to ask what is the minimum number of points that an m-K_n-residual graph must contain. We easily prove that this number is $(m+1)n$ and that the only m-K_n-residual graph with this number of points is $(m+1)K_n$. The same question for connected m-K_n-residual graphs is more interesting. We are able to show that a connected K_n-residual graph must have at least $2n+2$ points if $n \neq 2$. Furthermore, the cartesian product $K_{n+1} \times K_2$ is the only such graph with $2n+2$ points for $n \neq 2, 3, 4$. We complete the result by determining all connected K_n-residual graphs of minimal order for $n = 2, 3, 4$.

Although we have not obtained the minimum number of points for a connected m-K_n-residual graph, we include some canonical examples which might be expected to have smallest order when n is large.

In general the notation follows that of [1]. In particular $p(G)$ is the number of points in a graph G, $N(u)$ is the neighborhood of a point u consisting of all points adjacent to u. $N^*(u)$ is the closed neighborhood of u. Also, for any real x, the symbol $\lceil x \rceil$ denotes the ceiling of x defined as the smallest integer $n \geq x$.

2. Residually-complete graphs of minimum order

We begin this section with a simple observation which will turn out to be extremely useful.

Remark 1. If G is F-residual, then for any point u in G, the degree $d(u) = p(G) - p(F) - 1$. Hence every F-residual graph is regular, though this is generally not true for multiply-F-residual graphs (see Example 3).

Theorem 1. *Every m-K_n-residual graph has at least $(m+1)n$ points, and $(m+1)K_n$ is the only m-K_n-residual graph with $(m+1)n$ points.*

Proof. Let G be K_n-residual, and u, v nonadjacent points in G. Then $H_1 = G - N^*(u)$ and $H_2 = G - N^*(v)$ are disjoint copies of K_n contained in G, so $p(G) \geq 2n$. If $p(G) = 2n$, then $G = H_1 \cup H_2$ so all that remains to be shown is that there are no lines between H_1 and H_2, which is clear since G is $(n-1)$-regular by Remark 1.

Using induction on m, the rest of the theorem can easily be proved by similar arguments.

Theorem 2. *Every connected K_n-residual graph has at least $2n+2$ points if $n \neq 2$.*

The proof of this theorem requires a few preliminary results. We begin with the following definition.

For two points u, v in G, we say u is *K_n-adjacent* to v if there exists a copy of K_n in G which contains both u and v.

Lemma 2a. *Let G be a K_n-residual graph with $p(G) < 2n + \lceil \frac{1}{2}n \rceil$, and let u, v, w be points in G such that u is K_n-adjacent to v and v is K_n-adjacent to w. Then u is adjacent to w, in fact, u is K_n-adjacent to w.*

Proof. Let H_1 and H_2 be copies of K_n contained in G with $u, v \in H_1$ and $v, w \in H_2$. Suppose u is not adjacent to w. Then $w \in H_3 = G - N^*(u)$ which is another copy of K_n in G. Clearly $H_1 \cap H_3 = \emptyset$ since $H_1 \subset N^*(u)$. Thus $p(H_2 - H_3) \geq p(H_2 \cap H_1)$ and we see that $p(H_1 - H_2) + p(H_2 - H_3) \geq p(H_1) = n$. This shows that $\max\{p(H_1 - H_2), p(H_2 - H_3)\} \geq \lceil \frac{1}{2}n \rceil$. Now consider the degrees of v and w. We have

$$d(v) \geq p(H_2) - 1 + p(H_1 - H_2) = n - 1 + p(H_1 - H_2)$$

$$d(w) \geqslant p(H_3) - 1 + p(H_2 - H_3) = n - 1 + p(H_2 - H_3).$$

Hence there exists a point y in G with $d(y) \geqslant n - 1 + \lceil \frac{1}{2}n \rceil$, showing that

$$p(G) \geqslant n + (n - 1 + \lceil \tfrac{1}{2}n \rceil) + 1 = 2n + \lceil \tfrac{1}{2}n \rceil$$

by Remark 1, which contradicts the hypothesis $p(G) < 2n - \lceil \frac{1}{2}n \rceil$. Thus we see that u is adjacent to w. By repeating this argument, it is clear that u is adjacent to every point in H_2, and hence u is K_n-adjacent to w.

Remark 2. If G is a K_n-residual graph with $p(G) < 2n + \lceil \frac{1}{2}n \rceil$, then for any two nondisjoint copies H_1 and H_2 of K_n contained in G, we have $H_1 \cup H_2 \cong K_s$ where $s = p(H_1 \cup H_2)$.

Proof. Choose $v \in H_1 \cap H_2$, and let u, w be any two points in $H_1 \cup H_2$. Clearly u is K_n-adjacent to v and v is K_n-adjacent to w, so by Lemma 2a. u and w are adjacent.

Lemma 2b. *If G is a connected K_n-residual graph with $p(G) < 2n - \lceil \frac{1}{2}n \rceil$, then G contains a copy of K_{n+1}.*

Proof. Since G is connected and K_n-residual, by Theorem 1 we have $p(G) \geqslant 2n + 1$. Choose some copy of K_n in G, denoted by H_1, and let u be a point in H_1. Since $p(G) \geqslant 2n + 1$, we have $d(u) \geqslant n$ and thus we can find $v \in N^*(u) - H_1$. If $\langle H_1 \cup \{v\} \rangle \cong K_{n+1}$ we are done, so assume there exists $w \in H_1 - N^*(v)$. let $H_2 = G - N^*(v)$. Now H_1 and H_2 are nondisjoint copies of K_n in G, so $\langle H_1 \cup H_2 \rangle \cong K_s$ where $s = p(H_1 \cup H_2) \geqslant n + 1$ since $u \in H_1 - H_2$.

We are now ready to prove Theorem 2. Let G be a connected K_n-residual graph. The case where $n = 1$ is obvious since neither of the connected graphs of order 3, P_3 and K_3, is K_1-residual. Thus we assume $n \geqslant 3$. If $p(G) \geqslant 2n + \lceil \frac{1}{2}n \rceil$ we are done since $\lceil \frac{1}{2}n \rceil \geqslant 2$. If $p(G) < 2n + \lceil \frac{1}{2}n \rceil$, then G contains a copy of K_{n+1} which we denote by H. Since G is connected and $G - H \neq \emptyset$, we must have $d(u) \geqslant n + 1$ for some point u in H, and thus

$$p(G) \geqslant n + (n + 1) + 1 = 2n + 2$$

by Remark 1.

The next result determines the connected K_n-residual graphs of minimum order. It is interesting to note that for $n \neq 3, 4$ the graph is unique.

Theorem 3. *If $n \neq 2$, then $K_{n+1} \times K_2$ is a connected K_n-residual graph of minimum order, and except for $n = 3$ and $n = 4$, it is the only such graph. For each of the cases*

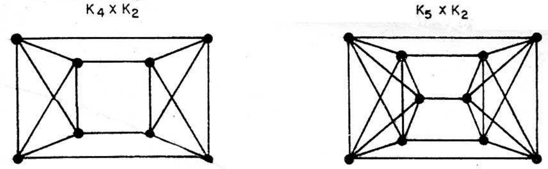

Fig. 1. Two examples of $K_{n+1} \times K_2$.

$n = 3$ and $n = 4$ there is exactly one other such graph. Finally, C_5 is the only connected K_2-residual graph of minimum order.

The graphs $K_4 \times K_2$ and $K_5 \times K_2$ are shown in Fig. 1 while the other smallest connected K_n-residual graphs for $n = 3$ and 4 are given in Figs. 2 and 3.

Proof. It is easy to verify that $K_{n+1} \times K_2$ is a connected K_n-residual graph for any n. Since $p(K_{n+1} \times K_2) = 2n+2$, Theorem 2 shows that $K_{n+1} \times K_2$ has minimum order for $n \neq 2$. Suppose $n \geq 5$ and that G is a connected K_n-residual graph with $p(G) = 2n+2$. Then $p(G) < 2n + \lceil \frac{1}{2}n \rceil$ so G contains a copy of K_{n+1}, which we denote by $L = \langle x_1, \ldots, x_{n+1} \rangle$. Since $d(x_i) = n+1$, it follows that $N^*(x_i) - L = \{y_i\}$. Also $G = \bigcup_{i=1}^{n} N^*(x_i)$ since otherwise we would have $L \subset G - N^*(u)$ for some point u in G. This shows that $G - L = \langle y_1, \ldots, y_{n+1} \rangle$ and since $p(G-L) = n+1$ we find that the y_i's are distinct. Moreover, for $i \neq j$ we see that $y_i, y_j \in G - N^*(x_k)$ for any $k \neq i, j$ and hence y_i and y_j are adjacent, showing that $G - L \cong K_{n+1}$. Clearly $G \cong K_{n+1} \times K_2$.

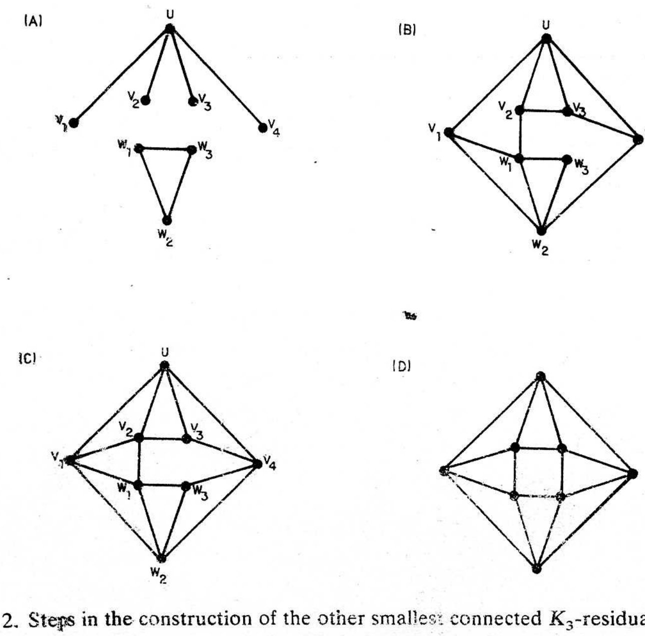

Fig. 2. Steps in the construction of the other smallest connected K_3-residual graph.

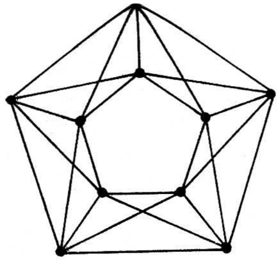

Fig. 3. The other smallest connected K_4-residual graph.

We now prove the remainder of the theorem involving the small cases $n \leq 4$. For $n=1, K_2 \times K_2 = C_4$ is the only regular connected graph of degree 2 on 4 points, and similarly for $n=2$, C_5 is the only regular connected graph of degree 2 on 5 points.

For $n=3$, suppose G is a connected K_3-residual graph with $p(G)=8$. If G contains a copy of K_4, then the same proof as for $n \geq 5$ will show that $G \cong K_4 \times K_2$. Thus we may assume that G does not contain a copy of K_4. Let

$$V(G) = \{u, v_1, v_2, v_3, v_4, w_1, w_2, w_3\}$$

where $N(u) = \{v_1, v_2, v_3, v_4\}$ and $\langle W \rangle = \langle w_1, w_2, w_3 \rangle \cong K_3$ (see Fig. 2a). Since $K_4 \not\subset G$ we see that $W \not\subset N(v_i)$ for any i, and for the same reason $N(w_i) \cap N(u) \neq N(w_j) \cap N(u)$ if $i \neq j$. Thus for each pair of distinct i and j we have

$$p(N(w_i) \cap N(u) \cap N(w_j)) = 1.$$

By symmetry we may assume that $N(w_1) = \{v_1, v_2, w_2, w_3\}$ and $N(w_2) = \{v_1, v_4, w_1, w_3\}$. These imply $\langle u, v_3, v_4 \rangle \cong K_3$ and $\langle u, v_2, v_3 \rangle \cong K_3$. In particular v_3 is adjacent to v_4 and v_2 is adjacent to v_3 (see Fig. 2b). Since $W \not\subset N(v_1)$, v_1 is not adjacent to w_3, hence either v_2 or v_4 is adjacent to w_3, and by symmetry we may assume v_4 is adjacent to w_3. Thus $N(v_4) = \{u, v_3, w_2, w_3\}$ so $\langle v_1, v_2, w_1 \rangle \cong K_3$ and in particular v_1 is adjacent to v_2 (see Fig. 2c). Finally, since $N(v_1) = \{u, v_2, w_1, w_2\}$ we have $\langle v_3, v_4, w_3 \rangle \cong K_3$ so v_3 is adjacent to w_3. Now every point in G has degree 4, so the construction is finished (see Fig. 2d).

For $n=4$, suppose G is a connected K_4-residual graph with $p(G)=10$. If G contains a copy of K_5, then as before one finds $G \cong K_5 \times K_2$. If G does not contain a copy of K_5, then similar arguments as for the case $n=3$ will construct the graph shown in Fig. 3.

3. Multiply-K_n-residual graphs

In this section we first note that for any m and n there are infinitely many connected m-K_n-residual graphs, then exhibit some canonical examples, and close with some conjectures on the minimum number of points in a connected m-K_n-residual graph.

Remark 3. For any choice of positive integers m and n, there are infinitely many connected m-K_n-residual graphs.

Proof. Observe that if G_1 and G_2 are disjoint m-K_n-residual graphs, then their join G_1+G_2 (as in [1, p. 21]) is a connected m-K_n-residual graph. Since $(m+1)K_n$ is an m-K_n-residual graph, we can repeatedly use the above technique to construct an infinite collection of graphs.

It is easy to see that G is K_n-residual if and only if $\bar{G}$ is n-regular and contains no triangles. This observation of R.W. Robinson verifies Remark 3 at once for $m=1$.

Example 1. The join $(m+1)K_n+(m+1)K_n$ is a connected m-K_n-residual graph with $2n(m+1)$ points.

Example 2. The cartesian product $K_{n+m}\times K_{m+1}$ is a connected m-K_n-residual graph with $(n+m)(m+1)$ points. This is easily proved by induction on m since we have already noted that $K_{n+1}\times K_2$ is K_n-residual.

Notice that for $n=m$, the graphs of Examples 1 and 2 have the same order although they are not isomorphic unless $n=1$.

Example 3. For each $m\geq 1$, the graph G_m defined by

$$V(G_m)=\{u_0,\ldots,u_{m+1},\ v_1,\ldots,v_m,\ w_0,\ldots,w_{m-1}\}$$

and

$$E(G_m)=\{u_iu_{i+1},\ u_iw_i,\ u_iv_{i-1},\ v_iw_i,\ v_iw_{i-1}\}$$

can be shown to be a connected m-K_2-residual graph with $3m+2$ points. The graphs G_m for $m=1,2,3,4$ are shown in Fig. 4, as well as another connected 3-K_2-residual with 11 points. Notice that the graph G_m is not regular unless $m=1$.

4. Unsolved problems and conjectures

We have only determined the minimum order of the connected K_n-residual graphs. The question is open for m-K_n-residual graphs when $m\geq 2$.

Conjecture 1. *If $n\neq 2$, then every connected m-K_n-residual graph has at least $\min\{2n(m+1),(n+m)(m+1)\}$ points.*

Every connected m-K_2-residual graph has at least $3m+2$ points.

Note that this quantity agrees with that of Theorem 2 for $m=1$ when $n\neq 2$, and with Theorem 3 when $n=2$.

We believe that there will be an analogous uniqueness result for $m\geq 2$.

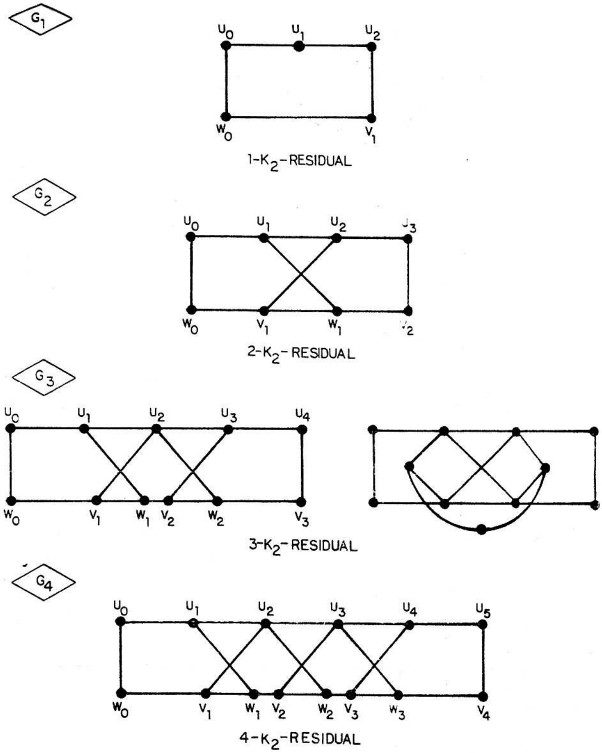

Fig. 4. Multiply-K_2-residual graphs of small order.

Conjecture 2. *For n large, there is a unique smallest connected m-K_n-residual graph.*

The *link* of a point u of a graph G, written $L(u)$, is the subgraph $\langle N(u)\rangle$ induced by the neighborhood of u. A graph G has *constant link* if for all $u, v \in V(G)$, $L(u) \cong L(v)$. Clearly G is K_n-residual if and only if its complement $\bar{G}$ has constant link $\bar{K}_n$.

In general, then, G is an F-residual graph if and only if $\bar{G}$ has constant link $\bar{F}$. In later communications we propose to investigate F-residual graphs for $F = K_n$, in order to determine the minimum order among such graphs, and to specify the corresponding extremal graphs.

References

[1] F. Harary, Graph Theory. (Addison-Wesley, Reading, MA, 1969).