On the Möbius function

By *P. Erdös* at Budapest and *R. R. Hall* at York

Introduction. In this paper we prove some results about the function

$$M(n, T) = \sum \{\mu(d) : d | n, d \leqq T\}.$$

Let $\omega(n)$ denote the number of distinct prime factors of n. If we split the divisors of n into Sperner chains, and note that the contribution to $|M(n, T)|$ from each chain is at most 1, we have

$$|M(n, T)| \leqq \binom{\omega(n)}{[\omega(n)/2]} = o(2^{\omega(n)}),$$

moreover the inequality is best possible. If $n = p_1 p_2 \cdots p_\omega$ where $p_{r+1} > p_1 p_2 \cdots p_r$ for every r then

$$-1 \leqq M(n, T) \leqq 1$$

for every T.

For almost all n, it is known [2] that

$$\max_T |M(n, T)| < A^{\omega(n)}$$

for any fixed $A > 3/e$. We do not know if the constant $3/e$ is sharp: maybe $A > 1$ is sufficient. It seems certain that for almost all n the innocent looking inequality

$$\max_T |M(n, T)| \geqq 2$$

holds, but we are unable to prove it.

Theorem 1. *For every $\varepsilon > 0$, there exists a T_0 such that for fixed $T > T_0$, the density of the integers n such that $M(n, T) \neq 0$ does not exceed ε. More precisely, this density is $\ll (\log T)^{-\gamma_0}$ where $\gamma_0 = 1 - (e/2)\log 2$.*

This result suggests that in some suitable sense, $M(n, T)$ is usually zero. One of us conjectured that for almost all n, we have

$$\sum \left\{\frac{1}{T} : T \leqq n,\ M(n, T) \neq 0\right\} = o(\log n),$$

and this is a corollary, with quite a lot to spare, of the following result.

0075-4102/80/0315-0010$02.00

Theorem 2. *Let q be fixed, $q \geqq 2$, $u = q/q-1$ and $\beta = 0$ or 1 according as $q > 2$ or not. Then for almost all n, we have*

$$\sum_{m \leqq n} \frac{1}{m} |M(n, m)|^q \leqq \psi(n) \{F(u)\}^{(q-1)\omega(n)} (\log\log n)^\beta$$

provided $\psi(n) \to \infty$ as $n \to \infty$. Here

$$F(u) = \frac{1}{2\pi} \int_0^{2\pi} |1 - e^{i\theta}|^u \, d\theta = \frac{2^u}{\sqrt{\pi}} \frac{\Gamma((u+1)/2)}{\Gamma((u+2)/2)}.$$

Remarks. Since

$$F(u) = \frac{1}{2\pi} \int_0^{2\pi} \left(2 \sin \frac{\theta}{2}\right)^u d\theta \geqq \frac{1}{2\pi} \int_0^{2\pi} 2^{u-1} (1 - \cos\theta) \, d\theta = 2^{u-1}$$

we have $\{F(u)\}^{(q-1)} \geqq 2$ with equality if and only if $q = 2$. In particular

$$\sum_{m \leqq n} \frac{1}{m} |M(n, m)|^2 \leqq \psi(n) 2^{\omega(n)} \log\log n$$

for almost all n. Since the normal order of $\omega(n)$ is $\log\log n$, the corollary mentioned above follows.

Next, if $\{d_i, 1 \leqq i \leqq 2^{\omega(n)}\}$ are the squarefree divisors of n arranged in increasing order, since $M(n, m) = M(n, d_i)$ for $d_i \leqq m < d_{i+1}$ we deduce that

$$\sum_i |M(n, d_i)|^2 \log \frac{d_{i+1}}{d_i} \leqq \psi(n) 2^{\omega(n)} \log\log n.$$

An immediate corollary is that

$$\min_i \log \frac{d_{i+1}}{d_i} \leqq \left(\sum_i |M(n, d_i)|^2\right)^{-1} \psi(n) 2^{\omega(n)} \log\log n.$$

Plainly

(1) $$\sum_i |M(n, d_i)|^2 \geqq 2^{\omega(n)-1}$$

(since $|M(n, d_i)|$ jumps ± 1 for every i). An old conjecture of Erdös [1] is that almost all integers have two divisors d, d' such that $d < d' < 2d$; this would follow from a small improvement of the above inequality (1).

Lemma 1. *Let $\delta(T)$ denote the asymptotic density of the integers n with at least one divisor d in the interval $[T, 2T]$. Then*

$$\delta(T) \ll (\log T)^{-\alpha} \quad \text{where } \alpha = 1 - \frac{1}{\log 2}\left(1 - \log \frac{1}{\log 2}\right).$$

Proof. Split the integers into two classes according as $\Omega_T(n) \leqq \kappa \log\log T$ or not, where Ω_T counts the prime factors $\leqq T$ of n according to multiplicity, and κ is to be chosen. For any $y \leqq 1$, the number of integers $\leqq x$ in the first class is

$$\leqq y^{-\kappa \log\log T} \sum_{n \leqq x}{}' y^{\Omega_T(n)}$$

where the dash denotes that n has a divisor in $[T, 2T]$. This is

$$\leqq (\log T)^{-\kappa \log y} \sum_{T \leqq d \leqq 2T} y^{\Omega_T(d)} \sum_{m \leqq x/T} y^{\Omega_T(m)}.$$

Plainly d has at most one prime factor $> T$. So this is

$$\ll y^{-1} (\log T)^{-\kappa \log y} \sum_{T \leqq d \leqq 2T} y^{\Omega(d)} \frac{x}{T} (\log T)^{y-1} \ll x y^{-1} (\log T)^{2y-2-\kappa \log y}.$$

We choose $y = \kappa/2$, which is in order provided $\kappa \leqq 2$. Hence the number of these integers does not exceed

$$x\kappa^{-1} (\log T)^{\kappa - 2 + \kappa \log 2/\kappa}.$$

The number of class 2 integers up to x does not exceed

$$z^{-\kappa \log\log T} \sum_{n \leqq x} z^{\Omega_T(n)}$$

provided $z \geqq 1$. This is

$$\ll x (\log T)^{z-1-\kappa \log z}$$

and we choose $z = \kappa$, so that we have to have $\kappa \in [1, 2]$. In fact we put $\kappa = 1/\log 2$ so that

$$\kappa - 2 + \kappa \log 2/\kappa = \kappa - 1 - \kappa \log \kappa = -\alpha.$$

This completes the proof.

Lemma 1′. *Let $\delta(T, \gamma)$ denote the asymptotic density of the integers n with at least one divisor d in the interval $[T, T \exp((\log T)^\gamma)]$. Then for $0 \leqq \gamma < 1 - \log 2$, we have*

$$\delta(T, \gamma) \ll (\log T)^{-\alpha(\gamma)} \quad \text{where } \alpha(\gamma) = 1 - \frac{1-\gamma}{\log 2}\left(1 - \log \frac{1-\gamma}{\log 2}\right).$$

In particular $\delta(T, \gamma) \to 0$ as $T \to \infty$ for each fixed γ in the range given.

Proof. We have

$$\delta(T, \gamma) \leqq \delta(T) + \delta(2T) + \cdots + \delta(2^r T) \quad \text{where } r = \left[\frac{(\log T)^\gamma}{\log 2}\right]$$

and by Lemma 1, we have

$$\delta(T, \gamma) \ll (\log T)^{\gamma - \alpha}.$$

This is insufficient. However, we notice that if we follow through the proof of Lemma 1, with the wider interval, the factor $(\log T)^\gamma$ only appears in the treatment of the integers in class 1, since the divisor property of the integers in class 2 was not used. Thus a different choice of κ, namely $\kappa = (1-\gamma)/\log 2$, is optimal in the new problem: and this gives the result stated.

Proof of Theorem 1. *Put* $H=\exp((\log T)^{\gamma})$. First of all, by Mertens' theorem, the density of integers with no prime factor $\leqq H$ is $\ll(\log T)^{-\gamma}$. Now consider an integer n with at least one prime factor $\leqq H$. Let $p_1=p_1(n)$ be the least prime factor of n and let $n=p_1^r m$, $p_1 \nmid m$. Then

$$M(n, T)=\sum_{\substack{d\mid n\\ d\leqq T}} \mu(d)=\sum_{\substack{d\mid m\\ d\leqq T}} \mu(d)+\sum_{\substack{d\mid m\\ p_1 d\leqq T}} \mu(p_1 d)=M(m, T)-M(m, T/p_1).$$

Hence $M(n, T)\neq 0$ implies that m, and so n, has at least one divisor in the interval $(T/H, T]$. By Lemma 1′, the density of such integers is $\ll(\log T)^{-\alpha(\gamma)}$. Therefore the density of integers for which $M(n, T)\neq 0$ is $\ll(\log T)^{-\gamma_0}$ where $\gamma_0=\alpha(\gamma_0)$, or $\gamma_0=1-\frac{e}{2}\log 2$. This is the result stated.

Lemma 2. *Uniformly for real, non-zero t, for $x\geqq\exp(1/|t|)$ and on any finite range $0<u_0\leqq u\leqq u_1$, we have that*

$$\sum_{p\leqq x} \frac{1}{p}|1-p^{it}|^u=F(u)\operatorname{loglog} x-F(u)\log^+\frac{1}{|t|}+O(\operatorname{loglog}(3+|t|)),$$

where

$$F(u)=\frac{1}{2\pi}\int_0^{2\pi}|1-e^{i\theta}|^u d\theta=\frac{2^u}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1}{2}(1+u)\right)}{\Gamma\left(\frac{1}{2}(2+u)\right)}.$$

Proof. We may show as in [3] Lemma 4 that for any y in the range $2\leqq y\leqq x$, we have

$$\sum_{y<p\leqq x}\frac{1}{p}|1-p^{it}|^u=F(u)\log\left(\frac{\log x}{\log y}\right)+O\left(\frac{1}{|t|\log y}+(3+|t|)e^{-\beta\sqrt{\log y}}\right)$$

where β is an absolute positive constant. If $|t|>1$, we choose y such that

$$\log y=\frac{1}{\beta^2}\log^2(3+|t|)$$

and make the trivial estimate

$$\sum_{p\leqq y}\frac{1}{p}|1-p^{it}|^u\ll\operatorname{loglog} y\ll\operatorname{loglog}(3+|t|).$$

If this $y>x$, we apply the trivial estimate to the whole sum. Next, if $|t|\leqq 1$, we set $\log y=1/|t|$. In this case $y\leqq x$ automatically. We have

$$\sum_{p\leqq y}\frac{1}{p}|1-p^{it}|^u\leqq\sum_{p\leqq y}\frac{1}{p}(|t|\log p)^u\ll(|t|\log y)^u=O(1)$$

and so we have

$$\sum_{p\leqq x}\frac{1}{p}|1-p^{it}|^u=F(u)\operatorname{loglog} x-F(u)\log\frac{1}{|t|}+O(1)$$

as required.

Proof of Theorem 2. For $n>1$, we have

$$f(n,t)=\sum_{d|n}\mu(d)d^{it}=-it\int_0^n M(n,z)z^{it-1}dz.$$

For $z\geqq n$, we have $M(n,z)=0$, so we can write

$$\frac{f(n,t)}{-it}=\int_{-\infty}^{\infty} M(n,e^s)e^{ist}ds.$$

We apply the Hausdorff-Young inequality for Fourier transforms, which gives

$$\left(\int_{-\infty}^{\infty}|M(n,e^s)|^q ds\right)^{1/q}\leqq\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}|t^{-1}f(n,t)|^u dt\right)^{1/u}$$

when $q\geqq 2$ and $u=q/q-1$. Since $M(n,z)=M(n,[z])$, we have

$$\sum_{m=1}^{n}\frac{1}{m}|M(n,m)|^q\leqq 2\int_0^{\infty}|M(n,z)|^q\frac{dz}{z},$$

and so if we define

$$\Delta(n,q)=\left(\sum_{m=1}^{n}\frac{1}{m}|M(n,m)|^q\right)^{1/q-1}$$

we have that

$$\Delta(n,q)\leqq 2\left(\int_{-\infty}^{\infty}|M(n,e^s)|^q ds\right)^{u/q}\leqq\frac{1}{\pi}\int_{-\infty}^{\infty}|t^{-1}f(n,t)|^u dt.$$

Let $\sum'$ denote summation restricted to integers n such that $\omega(n)>\frac{1}{2}\log\log x$. Then for $y>0$,

$$\sum_{n\leqq x}{}' y^{\omega(n)}\Delta(n,q)\leqq\frac{2}{\pi}\int_0^{\infty}\sum_{n\leqq x}y^{\omega(n)}|f(n,t)|^u t^{-u}dt,$$

since f is an even function of t. We split the range of integration according as $t\leqq 1/\log x$ or not: call these integrals I_1 and I_2. We consider I_2 first, and here we ignore the condition on $\omega(n)$. As in [3] Lemma 3 we have

$$\sum_{n\leqq x}y^{\omega(n)}|f(n,t)|^u\ll\frac{x}{\log x}\exp\left(y\sum_{p\leqq x}\frac{1}{p}|f(p,t)|^u\right)$$

uniformly for real t, and on any finite range $0\leqq u\leqq u_1$. We have $1\leqq u\leqq 2$, so we may apply lemma 2. We get

$$\sum_{n\leqq x}y^{\omega(n)}|f(n,t)|^u\ll\frac{x}{\log x}(t^*\log x)^{yF(u)}\log^{Ky}(3+t)$$

where K is an absolute constant and $t^*=t$ $(t\leqq 1)$, $t^*=1$ $(t>1)$. We restrict u and y by the conditions

(i) $u>1$,

(ii) $yF(u)-u\geqq -1$,

and we deduce that for fixed u and y,

$$I_2\ll x(\log x)^{yF(u)-1}(\log\log x)^{\beta}$$

where $\beta=0$ or 1, according as there is strict inequality in (ii) or not.

Let us set $y=1/F(u)$, so that we may replace the above conditions by

(iii) $1<u\leqq 2$ or $2\leqq q<\infty$.

We have $I_2 \ll x(\log\log x)^\beta$, where $\beta=0$ or 1 according as $u<2$, or not. Now consider I_1. By the arithmetic-geometric mean inequality, we have

$$|f(n,t)|\leqq \prod_{p|n} t\log p \leqq \left(\frac{t\log n}{\omega(n)}\right)^{\omega(n)}$$

and since we may assume $\omega(n)\geqq 2\geqq u$ (by the definition of $\sum'$) the integral is convergent. Indeed,

$$I_1 \leqq \sum_{n\leqq x}{}' \left(\frac{1/F(u)}{\omega(n)}\right)^{\omega(n)} (\log x)^{u-1} \ll x$$

since $F(u)\geqq 1$ and $\{\omega(n)\}^{\omega(n)}>\log x$ for large x. Putting these results together, we have now proved that for fixed $q\geqq 2$,

$$\sum_{n\leqq x}{}' \{F(u)\}^{-\omega(n)} \Delta(n,q) \ll x(\log\log x)^\beta .$$

Let $\psi_1(n)\to\infty$ as $n\to\infty$. For all but $o(x)$ integers n in this sum, we have

$$\Delta(n,q)\leqq \psi_1(n)\,\{F(u)\}^{\omega(n)}(\log\log n)^\beta$$

and so this is true for almost all n: the number of $n\leqq x$ neglected by $\sum'$ is $o(x)$, by the well known result of Hardy and Ramanujan that $\omega(n)$ has normal order $\log\log n$. Since $(q-1)\beta\geqq\beta$, we deduce that for almost all n,

$$\sum_{m\leqq n} \frac{1}{m}\,|M(n,m)|^q \leqq \psi(n)\,\{F(u)\}^{(q-1)\omega(n)}(\log\log n)^\beta ,$$

which is the result stated.

References

[1] *P. Erdös*, One the density of some sequences of integers, Bull. American Math. Soc. **54** (1948), 685—692.

[2] *R. R. Hall*, A problem of Erdös and Kátai, Mathematika **21** (1974), 110—113.

[3] *R. R. Hall*, Sums of imaginary powers of the divisors of integers, J. London Math. Soc. (2) **9** (1975), 571—580.

Mathematical Institute of the Hungarian Academy of Sciences, Budapest

Department of Mathematics, University of York, Heslington, York, Y01 5DD, England

Eingegangen 10. April 1979