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On the concentration of distribution of additive functions

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1. We say that $g(n)$ is *additive* if $g(mn)=g(m)+g(n)$ holds for every coprime pairs m, n of positive integers. If, moreover, $g(p^\alpha)=g(p)^\alpha$ for every prime power p^α, then $g(n)$ is called *strongly additive*. By $p, p_1, p_2, \ldots, q, q_1, q_2, \ldots$ we denote prime numbers, $c, c_1, c_2, \ldots$ are suitable positive constants. $P(n)$ and $\varkappa(n)$ denote the largest and the smallest prime factor of n. The symbol $\ll$ is used instead of O; $\#\{\ \}$ is the counting function of the set indicated in brackets $\{.\}$. For a distribution function $H(x)$ let $\varphi_H(\tau)$ denote its characteristic function. Let

$$Q(h)=Q_H(h)=\sup_x (H(x+h)-H(x))$$

be the continuity module — concentration — of H. We say that H satisfies a Lipschitz condition if $Q(h)\ll h$ as $h\to 0$.

We assume that $g(n)$ is strongly additive and that

$$\sum_p \frac{g^2(p)}{p}<\infty. \tag{1.1}$$

The theorem of Erdős—Wintner [1] guarantees that the function $g(n)-A_n$, where

$$A_n=\sum_{p<n}\frac{g(p)}{p}, \tag{1.3}$$

has a limit distribution, i.e. the relation

$$\frac{1}{N}\#\{n\leq N \mid g(n)-A_n<x\}\to F(x) \tag{1.4}$$

holds at every continuity point of $F(x)$, where $F(x)$ is a distribution function. If,

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moreover, $\Sigma g(p)/p$ converges, then the values $g(n)$ have a limit distribution too, i.e.

$$\frac{1}{N}\#\{n \leqq N \mid g(n) < x\} \to G(x), \tag{1.5}$$

at every continuity point of the distribution function $G(x)$.

We have the relations

$$\varphi_F(\tau) = \prod_p \left(\left(1-\frac{1}{p}\right)e^{-i\tau\frac{g(p)}{p}} + \frac{1}{p}e^{i\tau\left(1-\frac{1}{p}\right)g(p)}\right), \tag{1.6}$$

$$\varphi_G(\tau) = \prod_p \left(1-\frac{1}{p}+\frac{e^{i\tau g(p)}}{p}\right). \tag{1.7}$$

From these forms we can see that both F and G can be represented as the distribution of the sum of infinitely many mutually independent random variables having purely discrete distributions. By the well-known theorem of P. Lévy [2] G and F are continuous if

$$\sum_{p \in Z_g} 1/p = \infty, \quad \text{where } Z_g = \{p \mid g(p) \neq 0\}. \tag{1.8}$$

Furthermore, assuming the validity of (1.3) we have that F and G are of pure type, either absolutely continuous or singular (see E. Lukács [3]). To decide the question if a distribution function were absolutely continuous or singular seems to be quite difficult. The first result upon this has been achieved by P. Erdős [4]; namely it was proved that if $g(p)=O(p^{-\delta})$, δ being any positive constant, then $G(x)$ is singular. Recently Jogesh Babu [5] has proved that $G(x)$ is absolutely continuous if $g(n)$ is generated by $g(p)=(\log p)^{-a}$ $(0<a<2)$. The main idea of the proof is that $\varphi_G(\tau)$ is square-integrable in $(-\infty, \infty)$, and so by using Plancherel's theory of Fourier integrals it must have an inverse in $L^2(-\infty, \infty)$ that is the density function of $G(x)$.

It is known that a distribution function H satisfies Lipschitz condition if $|\varphi_H(\tau)|$ is integrable in $(-\infty, \infty)$, and so it is absolutely continuous. The method of Jogesh Babu gives that G satisfies Lipschitz condition if $g(p)=(\log p)^{-a}$ $(0<a<1)$.

The aim of this paper is to investigate the singularity or absolute continuouity of distribution functions for some classes of additive functions.

We shall prove the following theorems.

Theorem 1. *Let $g(n)$ be a strongly additive function,*

$$D(y) = \sum_{p>y} \frac{|g(p)|}{p}, \tag{1.9}$$

and suppose that the inequalities

$$D(t^A) < 1/t, \tag{1.10}$$

$$|g(p_1)-g(p_2)| > 1/t \quad \text{if} \quad p_1 \neq p_2 < t^\delta \tag{1.11}$$

hold, with suitable positive constants A and δ, for every large t. Then

(1.12) $$(\log t)^{-1} \ll Q_G(1/t) \ll (\log t)^{-1} \quad (t \to \infty),$$

where the constants involved by $\ll$ *may depend on g.*

This result was achieved by TJAN [7] and P. ERDŐS [8] for $\log \frac{\varphi(n)}{n}$, and for $\log \frac{\sigma(n)}{n}$, resp.

Theorem 2. *Let* $g(n)$ *be strongly additive satisfying* (1.1). *Then for the concentration* $Q(h)$ *of* $F(x)$ *or* $G(x)$ *(if it exists) we have*

(1.13) $$Q(4D_R) \geqq \frac{c}{\log R} \quad (R \geqq 2),$$

c being an absolute positive constant, and

(1.14) $$D_R = \left(\sum_{p>R} \frac{g^2(p)}{p} \right)^{1/2}.$$

Remarks.

1) This assertion is non-trivial only if $D_R \log R \to 0$ $(R \to \infty)$, since $Q_H(1/t) \gg 1/t$ $(t \to \infty)$ for every $H(x)$.
2) If $g(p) = (\log p)^{-\gamma}$ ($\gamma \geqq 1$ constant), then $D_R = (1+o(1)) \frac{(\log R)^{-\gamma}}{\sqrt{2\gamma}}$ and so $Q_G(1/t) \gg \frac{1}{t^{1/\gamma}}$.

Theorem 3. *If the strongly additive* $g(n)$ *is generated by* $g(p) = (\log p)^{-\gamma}$, *then*

(1.15) $$\frac{1}{t^{1/\gamma}} \ll Q_G(1/t) \ll \frac{(\log\log t)^2}{t^{1/\gamma}}$$

if $\gamma > 1$, *while for* $\gamma = 1$

(1.16) $$\frac{1}{t} \ll Q_G(1/t) \ll \frac{(\log\log t)^2 \log t}{t}.$$

Remarks.

1) We guess that $Q_G(1/t) \ll \frac{1}{t^{1/\gamma}}$ for $\gamma > 1$ but we are unable to prove it.
2) We also guess that $G(x)$ is singular if $0 \leqq g(p) \leqq (\log p)^{-\gamma}$, $\gamma > 2$. This seems not to be known even if $g(p) = (\log p)^{-\gamma}$.
3) By our method we could estimate the concentration for other functions if $g(p)$ is monotonic. The following assertion holds. Let $t(u) > 0$ to monotonically decreasing in $(1, \infty)$, $g(p) = t(p)$ for primes p. Let $y(\tau), z(\tau)$ be defined by the

relations $t(y(\tau))=\frac{y(\tau)^{1/4}}{\tau}$; $t(z(\tau))=1/\tau$. Suppose that for large τ, $y(\tau)<\tau^c$, $z(\tau)>e^{\tau^{1+\varepsilon}}$ ($\varepsilon>0$ constant), and that the integral

$$\int_{y(\tau)}^{z(\tau)} \frac{\cos \tau t(u)}{u \log u}\, du$$

is bounded as $\tau\to\infty$. Then $Q_F(h)\ll 1/h$. These conditions hold if $g(p)$ decreases regularly and

$$\sum \frac{g^2(p)}{p}<\infty, \quad \sum \frac{g(p)}{p}=\infty.$$

Theorem 4. *There exists a monotonically decreasing function $t(u)$ satisfying the conditions*

$$\sum \frac{t(p)}{p}=\infty, \quad \sum \frac{t^2(p)}{p}<\infty,$$

for which the distribution function $F(x)$ of the strongly additive $g(n)$ defined by $g(p)=$ $=t(p)$ is singular.

2. Proof of Theorems 2 and 4. We shall prove Theorem 2 for $F(x)$ only. The proof is almost the same for $G(x)$.

$F(x)$ can be represented as the distribution function of θ_1; $\theta_R=\sum_{p>R}\xi_p$, where ξ_p are mutually independent random variables with the distribution

$$P\left(\xi_p=g(p)\left(1-\frac{1}{p}\right)\right)=\frac{1}{p}, \quad P(\xi_p=-g(p)/p)=1-\frac{1}{p},$$

for the mean value $M\theta_R$ and variance $D\theta_R$ we have $M\theta_R=0$, $D\theta_R=D_R$. Consequently, by the Chebyshev inequality,

$$P(|\theta_R|<AD_R)\geqq 1-\frac{1}{A^2}.$$

So by

$$d=\sum_{p\leqq R}\frac{g(p)}{p}$$

we have

$$F(-d+AD_R)-F(-d-AD_R)\geqq P\left(\xi_p=\frac{-g(p)}{p}\,(\forall p\leqq R) A |\theta_R|\leqq AD_R)\right)$$

$$\geqq\left(1-\frac{1}{A^2}\right)\prod_{p\leqq R}(1-1/p)\gg(1-1/A^2)\cdot\frac{1}{\log R} \quad (R\geqq 2).$$

By putting $A=2$ our assertion follows immediately.

To prove Theorem 4 we define our $g(p)$ as follows. Let $R_1=1$, R_{l+1} be defined by $R_l=\log\log\log\log\log R_{l+1}$, $\lambda_l=\exp(\exp(\exp R_l))$, $g(p)=\frac{1}{\lambda_l}$ if $p\in[R_l, R_{l+1})$. Then

$$\sum_{p>R_l}\frac{g(p)}{p}=\infty, \quad \sum_{p>R_l}\frac{g^2(p)}{p}\ll\frac{1}{\lambda_l^2}\log\log R_{l+1}\ll\frac{1}{\lambda_l}.$$

Let m run over the square-free integers all prime factor of which is less than R. By Theorem 2, for fixed m the number of integers n with

$$n=mv\leqq N,\ \varkappa(m)\geqq R_l,\ g(v)-(A_N-A_{R_l})\in\left[-\frac{c}{\lambda_l}, \frac{c}{\lambda_l}\right]$$

is greater than a constant time of

$$\frac{N}{m}\prod_{p<R_l}\left(1-\frac{1}{p}\right).$$

Summing up for m we have

$$\#\left\{n=mv\leqq N\,|\,g(n)\in\bigcup_m\left[g(m)-A_{R_l}-\frac{c}{\lambda_l}, g(m)-A_{R_l}+\frac{c}{\lambda_l}\right]\right\}$$

$$\gg N\prod_{p<R_l}(1-1/p)\sum_{P(m)\leqq R_l}\frac{1}{m}\gg N.$$

So the intervals

$$\bigcup_m\left[g(m)-A_{R_l}-\frac{c}{\lambda_l}, g(m)-A_{R_l}+\frac{c}{\lambda_l}\right]$$

cover a positive percentage of integers. The whole length of these intervals is less than $c2^{\pi(R_l)}/\lambda_l$. This quantity tends to zero as $l\to\infty$. By this the theorem is proved.

3. Lemmas. Let $\mathcal{S}(A)$ be an arbitrary set of distinct square free integers m having the following properties:

(1) $A\leqq\varkappa(m)$,

(2) if $p_1|m_1$, $p_2|m_2$, $m_1\neq m_2\in\mathcal{S}(A)$, then $\frac{m_1}{p_1}\neq\frac{m_2}{p_2}$.

Let $\varrho(n)$ be a multiplicative function such that $0\leqq\varrho(p)\leqq 1+O(1/p^\delta)$ ($\delta>0$ constant). Moreover, let

$$T(A)=\sum_{m\in\mathcal{S}(A)}\frac{\varrho(m)}{m}. \tag{3.1}$$

Lemma 1. *For* $2\leqq A$ *we have*

$$T(A)\leqq\frac{c_1}{A\log A}, \tag{3.2}$$

c_1 *being an absolute constant.*

Proof. We split the elements of $\mathcal{S}(A)$ according to $P(m)\in[A^{2^h}, A^{2^{h+1}})$. Let $T_h(A)$ denote the part of the sum (3.1) corresponding to this interval. From (2) we have

$$T_h(A) \leqq \frac{1}{A^{2^h}} \sum \frac{\varrho(n)}{n}$$

where the sum extends over the square free n with $A \leqq \varkappa(n) < P(n) \leqq A^{2^{h+1}}$. So

$$\sum \frac{\varrho(m)}{m} \ll \prod_{A<p\leqq A^{2^{h+1}}} \left(1+\frac{\varrho(p)}{p}\right) \ll \frac{\log A^{2^{h+1}}}{\log A}.$$

Using this inequality for every $h \geqq 0$ we have (3.2).

Remark. Since $T(1) \leqq 1+T(2)$, therefore by Lemma 1, $T(1)$ is bounded.

We shall use the following Esseen type inequality due to A. S. Fainleib [6] which we quote as

Lemma 2. *For an arbitrary distribution function $H(x)$ we have*

$$Q_H(h) \leqq C \sup_{t\geqq 1/h} \frac{1}{t} \int_0^t |\varphi_H(\tau)|\, d\tau. \tag{3.3}$$

Lemma 3. *Let $\gamma>0$ be fixed,*

$$S = \sum_{\tau^{10}<p<e^{\tau^{1/\gamma}}} \frac{\cos \tau (\log p)^{-\gamma}}{p}. \tag{3.4}$$

Then S is bounded as $\tau\to\infty$.

Proof. First of all we shall prove that

$$E = \sum_{\tau^{10}\leqq n\leqq e^{\tau^{1/\gamma}}} \frac{\cos \tau (\log n)^{-\gamma}}{n \log n}$$

is bounded as $\tau\to\infty$. Indeed,

$$\left| E - \int_{\tau^{10}}^{e^{\tau^{1/\gamma}}} \frac{\cos \tau (\log u)^{-\gamma}}{u \log u}\, du \right| \ll \sum_{\tau^{10}\leqq n} \frac{1}{n \log n}\left(\frac{\tau}{(\log n)^\gamma} - \frac{\tau}{(\log (n+1))^\gamma}\right)$$

$$\ll \frac{\tau}{\tau^{10}(\log \tau)^{1+\gamma}}.$$

To estimate the integral we substitute $y=\tau/(\log u)^\gamma$, and we get immediately that

$$\int_{\tau^{10}}^{e^{\tau^{1/\gamma}}} \frac{\cos \tau (\log u)^{-\gamma}}{u \log u}\, du = \frac{1}{\gamma} \int_1^{\tau/(10 \log \tau)^\gamma} \frac{\cos y}{y^{1/\gamma}}\, dy = O(1).$$

So it is enough to prove that $S-E=O(1)$ as $\tau\to\infty$.

Let $\tau^{10}\leqq M\leqq e^{\tau^{1/\gamma}}$; $N_1=M+jN^{3/4}$ $(j=0,1,\dots,[M^{1/4}])$, $N=M^{3/4}$, $N_2=N_1+N$, and consider the quantity

$$S(N_1,N_2)=\sum_{N_1\leqq p<N_2}\frac{\cos\tau(\log p)^{-\gamma}}{p}-\sum_{N_1\leqq n<N_2}\frac{\cos\tau(\log n)^{-\gamma}}{n\log n}.$$

To estimate it we use the prime number theorem for short intervals in the form

$$\Delta_{N_1}(u)=\sum_{n=N_1}^{u}(\Lambda(n)-1)\ll\frac{N}{(\log N_1)^{10}}\quad(N_1\leqq u\leqq N_2). \tag{3.5}$$

Since

$$\frac{1}{N_1\log N_1}-\frac{1}{n\log n}=-\int_{N_1}^{n}\frac{\log x+1}{x^2(\log x)^2}\,dx\leqq\frac{2(n-N_1)}{N_1^2\log N_1}$$

for $N_1\leqq n\leqq N_2$, therefore

$$S(N_1,N_2)\ll\sum_{N_1\leqq p<N_2}1/p^2+\frac{N^2}{N_1^2}+\frac{|L(N_1,N_2)|}{N_1\log N_1}, \tag{3.6}$$

where

$$L(N_1,N_2)=\sum_{n=N_1}^{N_2}(\Lambda(n)-1)\cos\tau(\log n)^{-\gamma}.$$

By using partial summation,

$$L(N_1,N_2)=\Delta_{N_1}(N_2)\cos\tau(\log N_2)^{-\gamma}+\sum_{n=N_1}^{N_2-1}\Delta_{N_1}(n)\left(\cos\frac{\tau}{(\log n)^\gamma}-\cos\frac{\tau}{(\log(n+1))^\gamma}\right).$$

Hence, by (3.6) we get

$$L(N_1,N_2)\ll\frac{N}{(\log N_1)^{10}}\left(1+\sum_{n=N_1}^{N_2-1}\left|\cos\frac{\tau}{(\log n)^\gamma}-\cos\frac{\tau}{(\log(n+1))^\gamma}\right|\right).$$

Since $\tau/(\log n)^\gamma$ is monotonic and cosine satisfies Lipschitz condition, the last sum is majorated by

$$\frac{\tau}{\log N_1}-\frac{\tau}{\log N_2}.$$

Consequently,

$$\sum_{0\leqq j\leqq M^{1/4}}S(N_1,N_2)\ll\sum_{M\leqq p\leqq 2M}\frac{1}{p^2}+\frac{M^{3/2}M^{1/4}}{M^2}+\frac{1}{(\log M)^{11}}+$$
$$+\frac{M^{-1/4}}{(\log M)^{10}}\left(\frac{\tau}{\log M}-\frac{\tau}{\log 2M}\right).$$

By putting $M=2^h\tau^{10}$, $h=0,1,2,\dots,$ up to $M\leqq e^{\tau^{1/\gamma}}$ we have $S-E=O(1)$.

By this Lemma 3 has been proved.

4. Proof of Theorem 3. Let

$$\varphi(\tau)=\prod_p\left(1+\frac{e^{i\tau(\log p)^{-\gamma}}-1}{p}\right)$$

be the characteristic function of the limit distribution of $g(n)$ defined by $g(p)=(\log p)^{-\gamma}$. First we observe that

$$\log|\varphi(\tau)| \leq \mathrm{Re} \sum_{p \leq e^{\tau^{1/\gamma}}} \frac{e^{i\tau(\log p)^{-\gamma}}-1}{p} + O(1). \tag{4.1}$$

Lemma 3 and the relation

$$\sum_{p \leq y} \frac{1}{p} = \log\log y + O(1)$$

gives that

$$\log|\varphi(\tau)| \leq -\frac{1}{\gamma}\log\tau + O(1) + \mathrm{Re} \sum_{p \leq \tau^{10}} \frac{e^{i\tau(\log p)^{-\gamma}}}{p}. \tag{4.2}$$

Consequently, $\int_0^\infty |\varphi(\tau)| < \infty$ for $\gamma < 1$. Let $\gamma \geq 1$. From (4.2) we have

$$|\varphi(\tau)| \ll \tau^{-1/\gamma} |\psi(\tau)|, \tag{4.3}$$

where

$$\psi(\tau) = \prod_{p \leq R^{1/2}} \left(1 + \frac{e^{i\tau(\log p)^{-\gamma}}}{p}\right), \quad R \leq \tau \leq 2R. \tag{4.4}$$

Let $\psi(\tau) = \psi_1(\tau) \cdot \psi_2(\tau)$, where

$$\psi_1 = \prod_{p \leq (\log R)^4}, \quad \psi_2 = \prod_{(\log R)^4 < p \leq R^{1/2}}.$$

So we have

$$\int_R^{2R} |\varphi(\tau)|\, d\tau \ll \frac{1}{R^{1/\gamma}} (B_1(R) + B_2(R)), \tag{4.5}$$

where

$$B_j(R) = \int_R^{2R} |\psi_j(\tau)|^2 \, d\tau \quad (j = 1, 2). \tag{4.6}$$

First we estimate $B_2(R)$. We have

$$\psi_2(\tau) = 1 + \sum \frac{e^{i\tau g(m)}}{m},$$

where the summation is extended for the square-free m's satisfying $(\log R)^4 \leq \varkappa(m) \leq P(m) \leq R^{1/4}$. We have

$$B_2(R) \ll R + \sum \frac{1}{m} \min\left(R, \frac{1}{|g(m)|}\right) + \sum_{m,n} \frac{1}{mn} \min\left(R, \frac{1}{|g(m)-g(n)|}\right),$$

n runs over the same set as m.

Let

$$K(1/R) = \sup_x \sum_{g(m) \in [x, x+1/R)} 1/m. \tag{4.7}$$

Let x be fixed. We observe that the set of m's standing in the right hand side satisfies the conditions of Lemma 1 with $A=(\log R)^4$, $\varrho\equiv 1$. Indeed, if $|g(m_1)-g(m_2)|\leqq 1/R$, $p_1|m_1, p_2|m_2$, then

$$\left|g\left(\frac{m_1}{p_1}\right)-g\left(\frac{m_2}{p_2}\right)\right| \geqq |g(p_1)-g(p_2)|-|g(m_1)-g(m_2)| \geqq$$

$$\geqq \left|\frac{1}{(\log p_1)^\gamma}-\frac{1}{(\log p_2)^\gamma}\right|-1/R>0,$$

and so $\frac{m_1}{p_1}\neq\frac{m_2}{p_2}$. So we have

$$K(1/R)\ll\frac{\log\log R}{(\log R)^4}.$$

Furthermore, the contribution of the pairs m, n for which $|g(m)-g(n)|\geqq R^2$ is majorated by

$$\frac{1}{R^2}\prod_{p\leqq R^{1/2}}(1+1/p)^2\ll\frac{(\log R)^2}{R^2}.$$

Consequently

$$B_2(R)\ll R+\sum_n\frac{1}{n}\left(\sum_{0\leqq j\leqq R^3}\frac{R}{j+1}\left\{\sum_{|g(m)-g(n)|\in\left[j/R,\frac{j+1}{R}\right]}1/m\right\}\right)+ \tag{4.8}$$

$$+\sum_{0\leqq j\leqq R^3}\frac{R}{j+1}\left(\sum_{|g(m)|\in\left[j/R,\frac{j+1}{R}\right]}1/m\right)\ll R.$$

Since $|\psi_1(\tau)|\leqq\prod_{p\leqq(\log R)^4}(1+1/p)\ll\log\log R$, therefore $B_1(R)\ll(\log\log R)^2R$. So we have

$$\int_R^{2R}|\varphi(\tau)|\,d\tau\ll R^{1-1/\gamma}(\log\log R)^2.$$

Applying this inequality for $R=T/2^h$ $(h=1, 2, \ldots)$ we get

$$\frac{1}{T}\int_1^T|\varphi(\tau)|\,d\tau\ll\begin{cases}\dfrac{(\log\log T)^2}{T^{1/\gamma}}, & \text{if } \gamma>1,\\[2mm] \dfrac{(\log\log T)^2\log T}{T}, & \text{if } \gamma=1.\end{cases}$$

From Lemma 2 our theorem immediately follows.

5. Proof of Theorem 1. First we prove the second inequality in (1.12). Let

$$g(n;\,y)=\sum_{p|n,\ p\geqq y}g(p).$$

Since from (1.10)

$$\sum_{n\leqq N}|g(n;\,t^{2.4})|\leqq ND(t^{2.4})\leqq\frac{N}{t^2},$$

we have

$$\#\{n \leq N;\ |g(n;\ t^{2A})| \geq 1/t\} \leq \frac{N}{t}. \tag{5.1}$$

For a natural number n let $e(n)$ denote the product of those prime factors of n that are less than t^{2A}; let $f(n)=n/e(n)$. From (5.1) we get that with the exception of at most N/t integers if $n \leq N$ and $g(n) \in [x, x+1/t]$, then $g(e(n)) \in [x-1/t, x+1/t]$. Let x and t be fixed, and $a_1 < a_2 < \ldots < a_R$ be the sequence of those square-free integers all prime divisors of which is less than t^{2A} and $g(a_j) \in [x-1/t, x+1/t]$. Let $E(a_j)$ be the number of those $n \leq N$ for which $a_j | e(n)$ and $(a_j, e(n)) = a_j$ holds. By using the Eratosthenian sieve we have

$$E(a_j) \leq 1 + O(1) \frac{N\varrho(a_j)}{a_j} \prod_{p < t^{2A}} \left(1 - \frac{1}{p}\right) \quad (N \to \infty), \tag{5.2}$$

where $\varrho(m) = \prod_{p|m} \frac{1}{1-1/p}$. Since $\prod_{p<t^A} (1-1/p) \ll (\log t)^{-1}$,

we have

$$Q_G(1/t) \ll \frac{1}{t} + \frac{1}{\log t} \sup_x \sum_{g(a_j) \in [x-1/t, x+1/t]} \frac{\varrho(a_j)}{a_j}. \tag{5.3}$$

It has only remained to prove that

$$U_{x,t} = \sum_{g(a_j) \in [x, x+1/t]} \frac{\varrho(a_j)}{a_j} \ll 1 \tag{5.4}$$

uniformly for $x \in (-\infty, \infty)$ as $t \to \infty$.

We write every a_j as mv where $P(m) < t^\delta$, $\varkappa(v) \geq t^\delta$, or $v=1$. So

$$U_{x,t} = \sum_v \frac{\varrho(v)}{v} \left\{ \sum_{g(m) \in [x-g(v), x+1/t-g(v)]} \frac{\varrho(m)}{m} \right\}.$$

The set of m's satisfies the conditions of Lemma 1 (see (1.11)) so the inner sum is bounded, and we have

$$U_{x,t} \ll \prod_{t^\delta \leq p \leq t^{2A}} \left(1 + \frac{\varrho(p)}{p}\right) \ll 1.$$

We shall prove that

$$G(1/t) - G(-1/t) \geq \frac{c}{\log t} \quad (t \to \infty),$$

and by this the proof will be finished.

Let $P=\prod_{p<t^{c_1}} p$. It is obvious that

$$(5.5)\qquad \sum_{n\leq N,(n,P)=1} 1=((1+o(1))N\prod_{p<t^{c_1}}(1-1/p)\geq \frac{c_2N}{c_1\log t}\quad (N\to\infty),$$

c_2 is an absolute constant. Furthermore,

$$\sum_{n\leq N,(n,P)=1} |g(n)|\leq \sum_{q>t^{c_1}} |g(q)| \sum_{qm\leq N,(m,P)=1} 1\leq c_3N\prod_{p\leq t^{c_1}}\left(1-\frac{1}{p}\right)\sum_{q>t^{c_1}}\frac{|g(q)|}{q}.$$

By choosing $c_1=2A$, from (1.9) we have

$$\sum_{\substack{n\leq N,(n,P)=1\\ |g(n)|\geq 1/t}} 1\leq t\sum_{n\leq N,(n,P)=1} |g(n)|\leq tN\prod_{p|P}\left(1-\frac{1}{p}\right)\sum_{p>t^{2A}}\frac{|g(p)|}{p}\leq \frac{c_4N}{t\log t}.$$

This and (5.5) gives that

$$F(1/t)-F(-1/t)\geq \frac{c_2}{2A\log t}-\frac{c_4}{t\log t}\geq \frac{c_5}{\log t}.$$

By this the proof of our theorem is finished.

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