When the Cartesian Product of Directed Cycles is Hamiltonian

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ABSTRACT

The cartesian product of two hamiltonian graphs is always hamiltonian. For directed graphs, the analogous statement is false. We show that the cartesian product $C_{n_1} \times C_{n_2}$ of directed cycles is hamiltonian if and only if the greatest common divisor (g.c.d.) d of n_1 and n_2 is at least two and there exist positive integers d_1, d_2 so that $d_1 + d_2 = d$ and g.c.d. $(n_1, d_1) =$ g.c.d. $(n_2, d_2) = 1$. We also discuss some number-theoretic problems motivated by this result.

1. INTRODUCTION

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. The cartesian product (see p. 22 of [1]) of G_1 and G_2, denoted $G_1 \times G_2$, is the graph $G = (V, E)$ where $V = V_1 \times V_2$ and

$$E = \left\{ \{(u_1, v_1), (u_2, v_2)\} : \begin{array}{l} u_1 = u_2 \quad \text{and} \quad \{v_1, v_2\} \in E_2 \\ \text{or } v_1 = v_2 \quad \text{and} \quad \{u_1, u_2\} \in E_1 \end{array} \right\}$$

A graph $G = (V, E)$ is hamiltonian if there exists a listing $v_1, v_2, \ldots, v_n$ of the vertex set V so that $\{v_i, v_{i+1}\} \in E$ for $i = 1, 2, \ldots, n-1$ and $\{v_n, v_1\} \in E$. It is elementary to show that if G_1 and G_2 are hamiltonian, so is $G_1 \times G_2$.

Now let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be directed graphs, i.e, E_1 and

Journal of Graph Theory, Vol. 2 (1978) 137–142
 0364-9024/78/0002-0137$01.00

E_2 are sets of ordered pairs of V_1 and V_2, respectively. The cartesian product $G_1 \times G_2$ is the directed graph $G = (V, E)$ where $V = V_1 \times V_2$ and

$$E = \left\{ ((u_1, v_1), (u_2, v_2)) : \begin{array}{ccc} & u_1 = u_2 \quad \text{and} \quad (v_1, v_2) \in E_2 \\ \text{or} & v_1 = v_2 \quad \text{and} \quad (u_1, u_2) \in E_1 \end{array} \right\}.$$

A directed graph $G = (V, E)$ is said to be hamiltonian if there exists a listing $v_1\, v_2, \ldots, v_n$ of V so that $(v_i, v_{i+1}) \in E$ for $i = 1, 2, \ldots, n-1$ and $(v_n, v_1) \in E$. As we shall see, the cartesian product of hamiltonian directed graphs need not be hamiltonian.

2. DIRECTED CYCLES

For an integer $n \geq 2$, let C_n be the directed graph with vertex set $\{0, 1, 2, \ldots, n-1\}$ and edge set $\{(i, i+1) : i = 0, 1, 2, \ldots, n-1 \pmod n\}$. In this section we will determine when the cartesian product $C_{n_1} \times C_{n_2}$ of directed cycles is hamiltonian. We begin by developing some necessary conditions. We suppose that $v_1, v_2, \ldots, v_{n_1 n_2}$ is a hamiltonian cycle in $C_{n_1} \times C_{n_2}$. Without loss of generality we may assume that $v_1 = (0, 0)$.

For an integer $n \geq 2$, we denote by Z_n the cyclic group of order n. We use the symbols $\{0, 1, 2, \ldots, n-1\}$ for the elements of Z_n with the operation being addition modulo n. We denote the direct sum of Z_{n_1} and Z_{n_2} by $Z_{n_1} \oplus Z_{n_2}$ and adopt the natural convention of using group notation for the elements of $C_{n_1} \times C_{n_2}$. In particular note that for each $i = 1, 2, \ldots, n_1 n_2$, either $v_{i+1} = v_i + (1, 0)$ or $v_{i+1} = v_i + (0, 1)$.

We let V denote the vertex set of $C_{n_1} \times C_{n_2}$ and then set

$$V_1 = \{v_i : v_{i+1} = v_i + (1, 0)\} \text{ and } V_2 = \{v_i : v_{i+1} = v_i + (0, 1)\}.$$

Note that V_1 and V_2 are nonempty and their union is V.

Lemma 1. $v \in V_1$ if and only if $v + (1, n_2 - 1) \in V_1$.

Proof. For each vertex $u \in V$, there are exactly two vertices $u_1, u_2 \in V$ for which $(u_1, u) \in E$ and $(u_2, u) \in E$, i.e.,

$$u = u_1 + (1, 0) = u_2 + (0, 1).$$

Now suppose $v \in V_1$ and let $v = v_i$; then $v_{i+1} = v_i + (1, 0)$. If $v + (1, n_2 - 1) \in V_2$ and $v + (1, n_2 - 1) = v_j$, then $i \neq j$ but $v_{i+1} = v_{j+1}$. This contradicts the assumption that $v_1, v_2, \ldots, v_{n_1 n_2}$ is a hamiltonian cycle in $C_{n_1} \times C_{n_2}$, i.e., each vertex appears exactly one time in this list.

On the other hand, if $v+(1, n_2-1) \in V_1$ and $v \in V_2$, then $v+(1,0)$ does not appear in the list since $v+(1,0)=v_{i+1}$ and $v_i = v+(1, n_2-1)$ require $v+(1, n_2-1) \in V_2$ whereas $v_i = v$ requires $v \in V_1$. ∎

Let $(a,b) \in Z_n \oplus Z_{n_2}$ and let $\langle (a,b) \rangle$ denote the subgroup generated by (a,b). Then the order of $\langle (a,b) \rangle$ is the least common multiple of the integers $O(a)$ and $O(b)$ which are the orders of a and b in Z_{n_1} and Z_{n_2}, respectively. Let $H = \langle (1, n_2-1) \rangle = \langle (n_1-1, 1) \rangle$. Then $|H| =$ l.c.m. $(n_1, n_2) = n_1 n_2/d$ where $d =$ g.c.d. (n_1, n_2). [l.c.m.—least common multiple; g.c.d.—greatest common divisor.]

It follows from Lemma 1 that V_1 and V_2 are both the union of distinct cosets of H. Since they are nonempty and disjoint, we see that the greatest common divisor d of n_1 and n_2 must be at least two. However, as we shall see, the condition that d be at least two is not sufficient.

Lemma 2. $v_i \in V_1$ if and only if $v_{i+d} \in V_1$.

Proof. Note that for each i, there exist e_1, e_2 so that $e_1+e_2=d$ and $v_{i+d} = v_i + (e_1, e_2)$. We now show that $\{(e_1, e_2) : e_1+e_2 = d\} \subset H$. It suffices to show that $(d, 0) \in H$. Choose integers q_1 and q_2 which satisfy the Diophantine equation $n_1 q_1 + n_2 q_2 = -d$. Then

$$\begin{aligned}(n_1 q_1 + d)(1, n_2-1) &= (-n_2 q_2)(1, n_2-1) \\ &= (n_1 q_1 + d, -n_2 q_2 n_2 + n_2 q_2) \\ &= (d, 0). \quad ∎\end{aligned}$$

Now let $v_{d+1} = v_1 + (d_1, d_2)$. Then $d_1 + d_2 = d$ and $d_1, d_2 > 0$ since neither V_1 nor V_2 is empty.

Lemma 3. The order of $\langle (d_1, d_2) \rangle$ in $Z_{n_1} \oplus Z_{n_2}$ is $n_1 n_2/d$.

Proof. Let $t =$ order $\langle (d_1, d_2) \rangle$. It follows from Lemma 2 that $v_{1+kd} = v_1 + k(d_1, d_2)$. Since $v_1 + t(d_1, d_2) = v_1$ and we visit exactly d vertices between v_i and v_{i+d}, we see that $td = n_1 n_2$, i.e., $t = n_1 n_2/d$. ∎

Lemma 4. g.c.d. $(n_1, d_1) =$ g.c.d. $(n_2, d_2) = 1$.

Proof. Suppose that there exists a prime p so that $p \mid d_1$ and $p \mid n_1$. Let $t_1 = O(d_1)$ and $t_2 = O(d_2)$ in Z_{n_1} and Z_{n_2}, respectively. Let us also suppose that $p \mid d_2$. Then $p \mid d$ and pn_2. Then the order of $\langle (d_1, d_2) \rangle$ in $Z_{n_1} \oplus Z_{n_2}$ is l.c.m. (t_1, t_2). Since $d_1 \cdot (n_1/p) = (d_1/p) \cdot n_1$ and t_1 is the least integer for which $n_1 \mid d_1 t_1$, we see that $t_1 \mid (n_1/p)$; similarly $t_2 \mid (n_2/p)$. Therefore

$$\text{l.c.m.}\ (t_1, t_2) \le \text{l.c.m.}\ (n_1/p, n_2/p) = (n_1 n_2)/(pd) < (n_1 n_2)/d$$

contradicting Lemma 3.

On the other hand, suppose that $p \nmid d_2$. Then $p \nmid d$ and $p \nmid n_2$. Then $t_1 \mid (n_1/p)$ and $t_2 \mid n_2$ and

$$\text{l.c.m.}\ (t_1, t_2) \le \text{l.c.m.}\ (n_1/p, n_2) = \frac{n_1 n_2}{pd} < \frac{n_1 n_2}{d},$$

again contradicting Lemma 3.

We conclude that g.c.d. $(n_1, d_1) = 1$ and dually we have g.c.d. $(n_2, d_2) = 1$. ∎

We are now ready to present the principal result of the paper.

Theorem 1. The cartesian product $C_{n_1} \times C_{n_2}$ of directed cycles is hamiltonian if and only if $d = \text{g.c.d.}(n_1, n_2) \ge 2$ and there exist positive integers d_1, d_2 so that $d_1 + d_2 = d$ and $\text{g.c.d.}(n_1, d_1) = \text{g.c.d.}(n_2, d_2) = 1$.

Proof. The necessity has been established by the preceding Lemmas. Sufficiency is established by constructing the hamiltonian cycle in the obvious fashion. Let $v_1 = (0, 0)$ and $v_{d+1} = (d_1, d_2)$. Then choose any directed path $v_1, v_2, \ldots, v_d, v_{d+1}$ between $(0, 0) = v_1$ and $(d_1, d_2) = v_{d+1}$; e.g., let $v_2 = (1, 0)$, $v_3 = (2, 0), \ldots, v_{d_1+1} = (d_1, 0)$, $v_{d_1+2} = (d_1, 1)$, $v_{d_1+3} = (d_1, 2), \ldots$. Then construct the remaining part of the cycle using (as required by Lemma 4) the rule $v_{i+d} = v_i + (d_1, d_2)$. It is straightforward to verify that the construction produces a hamiltonian cycle (see [2] for details).

Example 1. $C_{40} \times C_{56}$ is hamiltonian. In this case $d = 8$ and we may choose either (3, 5) or (7, 1) for (d_1, d_2). Note that there are then $\binom{8}{3} + \binom{8}{1}$ different hamiltonian cycles.

Example 2. Let $n_1 = 2^4 \cdot 5 \cdot 11$ and $n_2 = 2^4 \cdot 3 \cdot 7 \cdot 13$. Then $C_{n_1} \times C_{n_2}$ is not hamiltonian. This is the smallest example where $d \ge 2$ but the product is not hamiltonian since it is relatively easy to show that if g.c.d $(n_1, n_2) = d$ and $2 \le d \le 15$, then suitable d_1 and d_2 can always be found.

Klerlein [3] has shown that the Cayley color graph of the direct product of cyclic groups using the standard presentation is the cartesian product of directed cycles of appropriate orders. Theorem 1 can then be applied to determine when this Cayley color graph is hamiltonian.

3. SOME NUMBER THEORETIC RESULTS

Following [2], we say that an integer d is prime partitionable if there exist n_1, n_2 with $d = \text{g.c.d.}\ (n_1, n_2)$, so that for every $d_1, d_2 > 0$ with $d_1 + d_2 = d$ either g.c.d. $(n_1, d_1) \neq 1$ or g.c.d. $(n_2, d_2) \neq 1$. The first ten prime partionable numbers are 16, 22, 34, 36, 46, 52, 56, 64, 66, and 70. Note that all these values are even. In [2] it is asked whether infinitely many prime partionable number exist and whether there are any odd prime partionable numbers. We settle these questions in the affirmative.

Theorem 2. There exist infinitely many prime partionable numbers.

Proof. It follows from a theorem of Motohashi [4] that there exist infinitely many primes pairs p_1, p_2 with $p_1 > 3$ and $p_2 = 2p_1 + 1$. To see that for such primes, $d = p_1 + p_2$ is always prime partionable, let $n_1 = d \cdot p_1 \cdot p_2$ and n_2 be the product of d with all the primes other than p_1 and p_2 which are less than d. ∎

We also found several odd prime partionable numbers by computer search for solutions to Diophantine equations. These values are $d = 15$, 395; $d = 397$, 197; $d = 1{,}655{,}547$; $d = 2{,}107{,}997$; and $d = 2{,}969{,}667$.

Example 3. For the value $d = 15$, 395 let $n_1 = dp_1p_2p_3$ where $p_1 = 197$, $p_2 = 317$, and $p_3 = 359$. Then let n_2 be the product of d with all primes other than p_1, p_2, and p_3 which are less than d. Then observe that $d - p_1 = 48\, p_2$; $d - p_2 = 42\, p_3$; $d - p_3 = 84\, p_1$; and $d - 1 = 86\, p_1$. Furthermore each of the terms $p_1^2, p_2^2, p_3^2, p_1p_2, p_1p_3$, and p_2p_3 is larger than d. Thus d is prime partionable and $C_{n_1} \times C_{n_2}$ is not hamiltonian.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the assistance of Professor Carter Bays in the computer search for odd prime partionable numbers.

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