Creation in Mathematics, 11, 1978

Continuation from "Creation in Mathematics, 10, 1977"

Problems and Results in Combinatorial Analysis

by P. Erdős

Kneser made the following pretty conjecture: Let $|S| = 2n+k$ and define a graph $G_{n,k}$ as follows: It's vertices are the $\binom{2n+k}{n}$ n-tuples of S. Two vertices are joined iff the corresponding n-sets are disjoint. Denote by $K(G)$ the chromatic number of G. Kneser conjectured $K(G_{n,k}) = k+2$. [1] $K(G_{n,k}) \leq k+2$ is immediate but the opposite inequality seems to present great and unexpected difficulties. Szemerédi proved (unpublished) that $K(G_{n,k})$ tends to infinity uniformly in k. Hajnal and I and no doubt many others tried to attack this problem by the following extension of our theorem with Ko and Rado. Let $|S| = n = 2k+1$, $A_i \subset S$, $B_j \subset S$, $1 \leq i \leq t_1$, $1 \leq j \leq t_2$, the sets $A_1, \ldots; B_1, \ldots$ are all distinct, $A_{i_1} \cap A_{i_2}$, $1 \leq i_1 < i_2 \leq t_1$ and $B_{j_1} \cap B_{j_2}$, $1 \leq j_1 < j_2 \leq t_2$, are all non-empty. Is it true that

[1] The solution is in (The remarks of the Editor):
1. Juliusz Reichbach: Coloring and Kneser-Erdős Conjecture, Creation in Mathematics, 10, 1977
2. A. Schrijver: Vertex-critical subgraphs of Kneser-graphs, (Prepublication), Amsterdam, Mathematisch Centrum, Feb. 1978.

(6) $$t_1 + t_2 \leq \binom{n-1}{k-1} + \binom{n-2}{k-1} \quad ?$$

Equality in (6) if all the A's contain 1 and all the B's contain 2 elements.

Kneser's conjecture can be extended to r-graphs. Let $|S| = rn + k$. The vertices of our r-graph are the n-tuples of S. The edges are the sets

$$A_{i_1}, \ldots, A_{i_r}\ ;\ |A_{i_j}| = k,\ 1 \leq j \leq r,$$

and any two of the r k-sets are disjoint. - Then the chromatic number of this r r-graph should be $k+2$.

B. Grunbaum asked the following geometric question:
Let there be given n points in the plane, join any two of them by a line. What are the possible number of lines one gets? The number of lines is clearly at most $\binom{n}{2}$ and it can never be $\binom{n}{2} - 1$ and $\binom{n}{2} - 3$. I showed that there is an absolute constant c so that every $cn^{\frac{3}{?}} < t < \binom{n}{2} - 3$ can occur as the number of lines determined by an n-set. It follows from a result of Kelly and Moser that the order of magnitude $cn^{\frac{3}{?}}$ is best possible but the exact value of c is not known.

In this connection the following combinatorial problem in of interest:
Let $|S| = n$ and define I_r as a set of integers with the following property:
$t \in I_r$ iff there in a family of subsets $A_k \subset S$, $1 \leq k \leq t$ so that every r-tuple of S is contained in one and only one of the A's. Let us first in-

vestigate the $r = 2$. Clearly all integers in I_2 are $\leq \binom{r}{2}$, $1 \in I_2$, $\binom{n}{2} - 1$ and $\binom{n}{2} - 3$ not in I_2. A theorem of de Bruijn and myself states that no integer $1 < t < n$ is in I_2. Trivially $n \in I_2$ and also $\binom{n}{2} - 2 \in I_2$. I showed without much difficulty that there are absolute constants c_1 and c_2 so that every integer $n + c_1 n^{c_2} < t < \binom{n}{2} - 3$ is in I_2. It seems likely that $c_2 =$???. If $n = p^2 + p = 1$ (i.e. if there is a finite geometry) it is easy to see that every $p^2 + 2p + c\sqrt{p} = n + 2\sqrt{n} + c < t < \binom{n}{2} - 3$ belongs to I_2.

On the other hand A. Bruen recently proved that if $n = k$ then $t \in I_2$ if $k^2 < t < k^2 + k$.

It seems that the results of A. Bruen and Bridges will involve that there is an absolute constant $c > o$ so that for every n there is a t not in I_2 which is $> n + c\sqrt{n}$.

It was observed by Hanani that the smallest nontrivial value of I_3 is $cn^{\frac{3}{2}}$ and it follows from the existence of Möbius (or inversive) planes that I_3 contains all integers t, $(1 + c(???)n^{\frac{3}{2}} < t \leq \binom{n}{3}$ except the integers $\binom{n}{3} - i$ where i is not of the form $\sum_{j \geq 4} a_j \left(\binom{j}{3} - 1\right)$, $a_j \geq 0$.

For $r > 3$ it is much more difficult to get sharp results for I_r. It is easy to see that if $t > 1$, $t \in I_r$, then $t > cn^{\frac{r}{2}}$. This follows from the fact that not may of the sets A_k can be larger than $(1+\varepsilon)r^{\frac{1}{2}}n^{\frac{1}{2}}$, for otherwise $|A_i \cap A_j| > r$ (see e.g. Hylten-Cavallus, On a combinatorial problem, Colloq. Math. 6, 1958, 59-65). But it seems hard to prove that I_r contains an integer $1 < t < cn^{\frac{r}{2}}$.

The problem is to find $c_1 n^{\frac{r}{2}}$ sets A_k of size of the order of magnitude $n^{\frac{1}{2}}$ so that every r-tuple of our set $|S| = n$ should be contained in one and only one of the A_k's. Such construction is known for $r = 2$ and $r = 3$, but it is open for $r > 3$.

Before closing this chapter I state one of the many unsolved problems in our survey paper with Kleitman: - Let $|S| = n$, $A_i \subset S$, $1 \leq i \leq t$; assume that for no three distinct A's, $A_i \cap A_j = A_k$ or $A_i \cup A_j = A_k$. We conjectured that for even n max $t = \binom{n}{[\frac{n}{1}]} + 1$. Clements observed that this conjecture, if true, is best possible.

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The last part of the printed research in the next number of this journal. And afterwards every number of our journal "Creation in Mathematics" will contain a whole research of the regarded author.

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