On Products of Consecutive Integers

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1. Introduction

We define $A(n, k) = (n + k)!/n!$ where n, k are positive integers and we wish to examine divisibility properties of $A(n, k)$.

First observe that the cases $k = 1, 2$ are special. The relations

$$tA(n, 1) = A(m, 1) \qquad \text{and} \qquad tA(n, 2) = A(m, 2)$$

each have infinitely many solutions n, m—the first for every positive integer t; the second for every positive integer that is not a square, as can be seen from Pell's equation

$$(2m + 1)^2 - t(2n + 1)^2 = 1 - t.$$

On the other hand, it is well known from the Thue–Siegel theorem that for given $k \geq 3$ and fixed $t > 1$ the equation

$$tA(n, k) = A(m, k) \tag{1.1}$$

has only a finite number of solutions in integers n, m.

† Research of the second author was supported in part by Grant MCS 76-06988.

It is possible that (1.1) has only a finite number of solutions for fixed t and variable $k > 2$, but we cannot prove this even in the case $t = 2$ without additional hypotheses on n. Perhaps the following stronger conjecture holds: $A(n, k) = tA(m, l)$ has only a finite number of solutions $m \geq n + k$ for every rational t if $\min(k, l) > 1$, $\max(k, l) > 2$.

Let $f(n, k)$ be the least positive integer $m > n$ so that

$$A(m, k) \equiv 0 \pmod{A(n, k)}. \tag{1.2}$$

We obviously have $f(n, k) \leq A(n, k) - k$ and it is easy to see that for $k > 1$ we get several residue classes (in addition to $-k$) (mod $A(n, k)$) for m which ensure that (1.2) holds. The number of these residue classes is always larger than k, in fact exponential in k, so that the above inequality on $f(n, k)$ can always be improved.

The algebraic identities

$$x(x+1)(x+2) = (x^2+2x)(x+1)$$

$$\text{which divides } (x^2+2x)(x^2+2x+1)$$

$$x(x+1)(x+2)(x+3) = (x^2+3x)(x^2+3x+2)$$

$$x(x+1)(x+2)(x+3)(x+4) = (x^2+4x)(x^2+4x+3)(x+2)$$

$$\text{which divides } (x^2+4x)(x^2+4x+3)(x^2+4x+4)$$

show that $A(n, 3)$ divides $A(n^2+4n+1, 3)$; $A(n, 4)$ divides $A(n^2+5n+2, 4)$; and $A(n, 5)$ divides $A(n^2+6n+4, 5)$. Thus $f(n, 3) \leq n^2+4n+1$, $f(n, 4) \leq n^2+5n+2$, $f(n, 5) \leq n^2+6n+4$. It seems likely that these bounds are attained for infinitely many, perhaps for almost all, values of n. One might ask whether we get $f(n, k) < n^{k-\delta}$ for all (almost all) large n and some $\delta > 0$ when $k > 5$. In the other direction we would like to know whether $f(n, k) > n^{1+\delta}$ for infinitely many (almost all) n for some $\delta > 0$ when $k > 1$. For $k = 2$, we have infinitely many n with $f(n, k) \sim \sqrt{2n}$. Are there infinitely many n with $f(n, k) < cn$ for fixed c, $k > 2$?

A function closely related to $f(n, k)$ is $g(n, k)$, the minimal integral value $A(m, k)/A(n, k)$, $m > n$. The above discussion shows that $g(n, k) \sim f(n, k)^k/A(n, k)$ for fixed k and thus $g(n, k) \ll n^k$ for $k \leq 5$. Table 1 gives values of $f(n, k)$, $g(n, k)$ for small n and k.

One may try to estimate the density $d(n, k)$ of integers m for which (1.2) holds. Obviously $d(n, 1) = 1/(n+1)$. For $k = 2$ we have

$$d(n, 2) = 2^{\omega(A(n, 2))}/A(n, 2), \tag{1.3}$$

where $\omega(x)$ denotes the number of distinct prime factors of x. Relation (1.3) follows from the fact that $A(m, 2)$ is divisible by $A(n, 2)$ if and only if we can

TABLE 1

n \ k	1		2		3		4		5		6		7	
	f	g	f	g	f	g	f	g	f	g	f	g	f	g
1	3	2	2	2	3	5	2	3	4	21	2	4	3	15
2	5	2	7	6	3	2	6	14	4	6	3	3	5	22
3	7	2	14	12	7	6	4	2	11	78	6	11	13	646
4	9	2	8	3	12	13	6	3	13	68	22	1794	20	2691
5	11	2	13	5	13	10	52	2915	13	34	6	2	17	437
6	13	2	47	42	25	39	52	1749	17	57	16	969	7	2
7	15	2	62	56	53	231	52	1113	31	476	50	18921	21	345
8	17	2	34	14	42	86	32	119	50	2703	50	10812	20	138
9	19	2	43	18	19	7	51	477	51	1908	20	46	59	68076
10	21	2	31	8	63	160	62	720	51	1272	60	11346	49	11925
11	23	2	38	10	25	9	24	15	60	1891	46	1645	62	33902
12	25	2	76	33	62	96	61	372	47	420	50	1749	50	5247
13	27	2	19	2	47	35	31	22	31	42009	151	695981	169	11865205
14	29	2	79	27	117	413	268	72899	131	30954	284	20233213	149	3346915
15	31	2	254	240	269	4065	302	92415	319	1860516	284	14452295	169	5393275

factor $A(n, 2)$ into two relatively prime divisors $A(n, 2) = d_1 d_2$, $(d_1, d_2) = 1$, and require

$$m \equiv -1 \pmod{d_1}, \qquad m \equiv -2 \pmod{d_2}.$$

Since there are $2^{\omega(A(n,2))}$ such factorizations of $A(n, 2)$, we get that many residue classes (mod $A(n, 2)$) for m.

For $k > 2$, the problem of computing $d(n, k)$ becomes messier, but not intrinsically difficult. The number of residue classes (mod $A(n, k)$) to which m must belong remains $O(n^{\varepsilon})$ for every $\varepsilon > 0$ and hence

$$\frac{1}{n^k} \ll d(n, k) \ll \frac{1}{n^{k-\varepsilon}}$$

for all values of k.

Another question is that of determining those $m > n$ so that there exists some k for which (1.2) holds. Since for $k > m - n$, we have

$$\frac{A(m, k)}{A(n, k)} = \frac{A(n + k, m - n)}{A(n, m - n)},$$

which is certainly an integer for $k = A(n, m - n) - m$, the problem becomes trivial unless we restrict the values of k to $k \leq m - n$.

For $n = 1$ and any m we see that

$$\frac{A(m, p - 1)}{A(1, p - 1)} = \frac{1}{p}\binom{m + p - 1}{p - 1}$$

is divisible by p unless $m \equiv 0 \pmod p$. Since m cannot be divisible by all primes $\leq m$ except when $m = 2$, we see that for every $m > 2$ there exists a k, $1 \leq k \leq m - 1$ so that $A(1, k)$ divides $A(m, k)$. The question whether there exists a k, $1 \leq k \leq m - 2$, so that $A(2, k)$ divides $A(m, k)$, which is equivalent to $\binom{k+2}{2}$ divides $\binom{m+k}{k}$, seems much more difficult to decide. The general problem can be stated as follows:

Given $n > 1$ is it true that for all (almost all) large m there exists a k, $1 \leq k \leq m - n$ so that

$$\binom{k+n}{n} \bigg| \binom{m+k}{k}? \tag{1.4}$$

If not, what is the density $d^*(n)$ of integers m for which (1.4) has a solution with $1 \leq k \leq m - n$?

In Section 2 we show that for bounded ratios m/n and any $\delta > 0$ we get only a finite number of solutions of (1.2) with $k > \delta n$.

In Section 3 we treat the special case in which the set $\{n + 1, \ldots, n + k\}$ contains a prime and m/n is bounded to show that (1.2) has only a finite number of solutions $2 \leq k \leq m - n$ in that case. We give an example which may prove the only one with $m \leq 2n$. Finally, we mention some additional problems and conjectures.

2. The Case $k \geq \delta n$

In this section we prove the following.

Theorem 2.1 *Given positive numbers δ, Δ so that $k \geq \delta n$ and $n + k \leq m \leq \Delta n$, then there exists an $n_0 = n_0(\delta, \Delta)$ so that the congruence* (1.2) *has no solution with $n \geq n_0$.*

The proof depends on showing that for all large n there exists a prime p in the interval $[n + 1, n + k]$ that divides $A(n, k)$ to a higher power than it divides $A(m, k)$.

Lemma 2.2 *Assume that the hypotheses of Theorem* 2.1 *are satisfied and that every prime $p \in [n + 1, n + k]$ divides $A(m, k)$. Then for every $\varepsilon > 0$ there exists an $n_1 = n_1(\varepsilon)$ so that for all $n \geq n_1$ there exists an integer l, $2 \leq l < \Delta$, with*

$$(l - \varepsilon)n + (l - 1)k < m < lk + \varepsilon n \tag{2.1}$$

and hence

$$(l - 2\varepsilon)n < k. \tag{2.2}$$

Note that it is possible to have every prime that divides $A(n, k)$ also divide $A(m, k)$. This always happens when $k = ln$, $m = l^2 n$.

Proof Let n be so large that there exists a prime in $[n+1, n+k]$ and let p be the largest prime in that interval. Let l be the largest integer so that $lp \le m+k$. Then $l \ge 2$ and for large n we have $p > n+k-\delta n$. Hence

$$l \le \frac{m+k}{p} < \frac{m+k}{n+k-\delta n} < 1 + \frac{m-n}{(1+\delta-\delta)n} < 1 + \Delta - 1 = \Delta.$$

Now pick n so large that $p > n + k - \varepsilon n/\Delta$. Then

$$m + k \ge lp > ln + lk - \varepsilon n$$

and hence

$$(l-\varepsilon)n + (l-1)k < m.$$

Let q be the smallest prime so that $m + k < (l+1)q$. For large n we must have $n < q$ since otherwise l times every prime in $[n+1, n+k]$ would lie in $[m+1, m+k]$, which is impossible. Let n be so large that

$$\frac{m+k}{l+1} < q < \frac{m+k}{l+1} + \frac{\varepsilon n}{l(l+1)},$$

then $m < lq$ and hence

$$(l+1)m < l(m+k) + \varepsilon n$$

or

$$m < lk + \varepsilon n.$$

Lemma 2.3 *Assume that the hypotheses of Theorem* 2.1 *are satisfied and that* s *is an integer such that* $n < k/s$ *and every prime* $p \in [k/t, (n+k)/t]$, $t = 1, 2, \ldots, s$, *satisfies*

$$A(m, k) \equiv 0 \pmod{p^t}.$$

Assume further that for the integer l *in Lemma* 2.2 *we have*

$$m + \frac{a}{b}k - \varepsilon n < l(n+k) < m + \frac{c}{d}k + \varepsilon n \tag{2.3}$$

where a/b *and* c/d *are consecutive terms in the Farey series of order* s *and* ε *is a given number,* $0 < \varepsilon < s^{-2}$.

Then there exists an $n_2 = n_2(\varepsilon)$ *so that for all* $n \ge n_2$ *we have*

$$(l-\varepsilon)n + \left(l - \frac{c}{d}\right)k < m < \left(l - \frac{a}{b}\right)k + \varepsilon n \tag{2.4}$$

and hence

$$k > bd(l-2\varepsilon)n \ge s(l-2\varepsilon)n. \tag{2.5}$$

Proof For $s = 1$, this is Lemma 2.2. We now proceed by induction on s. If $\max\{b, d\} < s$, then the result follows from the induction hypothesis.

If $d = s > 1$, then $b < s$ and the first inequality in (2.4) follows directly from (2.3) while the second inequality holds by the induction hypothesis.

Now assume that $b = s$, $d < s$. Then the first inequality in (2.4) is still an immediate consequence of (2.3). If the second inequality were false, we would have

$$sl\frac{k}{s} < m + \frac{a}{s}k - \varepsilon n < sl\frac{n+k}{s}.$$

If $l(n + k) \le m + (a/s)k + \varepsilon n$ and n is sufficiently large, then there exists a prime p so that

$$m + \frac{a}{s}k - 2\varepsilon n < slp < m + \frac{a}{s}k + \varepsilon n \qquad \text{and} \qquad \frac{n+k}{s} - \frac{\varepsilon n}{sl} < p < \frac{n+k}{s}.$$

Hence $(sl - a)p \le m$ and $(sl + s - a)p > m + k$ and $A(m, k) \not\equiv 0 \pmod{p^s}$, contrary to hypothesis.

We may therefore assume that $l(n + k) > m + (a/s)k + \varepsilon n$; then for large n there exists a prime p with

$$m + \frac{a}{s}k < slp < m + \frac{a}{s}k + \frac{\varepsilon n}{sl} \qquad \text{and} \qquad \frac{k}{s} + \frac{\varepsilon n}{sl} < p < \frac{n+k}{s}.$$

Thus, again, $(sl - a)p \le m$ while $(sl + s - a)p > m + k$ and $A(m, k) \not\equiv 0 \pmod{p^s}$, contrary to hypothesis.

Proof of Theorem 2.1 If $l > 2$ and every prime $p \in [k/2, (n + k)/2]$ satisfies $A(m, k) \equiv 0 \pmod{p^2}$, then, according to (2.5), we have $k > 3n$. Now let s be the largest integer for which $sn < k$. If every prime $p \in [k/t, (n + k)/t]$, $t = 1, 2, \ldots, s$ satisfies $A(m, k) \equiv 0 \pmod{p^t}$, then, according to (2.5), we have $k > s(2 - 2\varepsilon)n > (s + 1)n$, a contradiction.

Now if $l = 2$ and $2n < k$, then Lemma 2.3 can be applied as before. If $l = 2$ and $2n \ge k$, then every prime $p \in [n + 1, (n + k)/2]$ satisfies $A(n, k) \equiv 0 \pmod{p^2}$. But, according to Lemma 2.2, we have

$$(4 - 3\varepsilon)n < m < m + k < (6 + \varepsilon)n.$$

Thus for large n there exists a prime p,

$$(\tfrac{5}{4} - \varepsilon)n < p < (\tfrac{5}{4} + \varepsilon)n,$$

so that for sufficiently small ε we have $3p < m$ while $5p > m + k$ and $A(m, k) \not\equiv 0 \pmod{p^2}$.

3. The Case $A(m, k) \equiv 0 \pmod{A(n, k)}$; $n + k \le m \le \Delta n$ and $\{n+1, \ldots, n+k\}$ Contains a Prime

We first mention the interesting example

$$A(32, 6) = 37 \cdot A(16, 6). \tag{3.1}$$

Here two of the integers 17, 18, 19, 20, 21, 22 are primes and 17 may well be the largest n which solves our problem in case $\Delta = 2$. In the following we show that we can find an effective bound for all solutions k, n, m.

From Theorem 2.1 we know that we can restrict attention to cases $k < \delta n$ where δ is any fixed positive number. Since there exists a prime p with

$$n + 1 \le p \le n + k \le m \le \Delta n \tag{3.2}$$

and $A(m, k)/A(n, k)$ is an integer, there must exist an integer l so that

$$m + 1 \le lp \le m + k. \tag{3.3}$$

Thus

$$ln + l - k \le m \le ln + (l-1)k. \tag{3.4}$$

Lemma 3.1 *Every integer*

$$x \in [n+1, n+k]\left[\frac{m+1}{l}, \frac{m+k}{l}\right]$$

has all prime divisors less than $(l+1)k$.

Proof Assume that x has a prime divisor $q > (l+1)k$. Now, either $lx < m + 1$ and $lx + q > ln + lk \ge m + k$ so that q does not divide $A(m, k)$, or $lx > m + k$ and $lx - q < ln + lk - (l+1)k < m$ and again q does not divide $A(m, k)$.

The set of integers in $[n+1, n+k]\backslash[(m+1)/l, (m+k)/l]$ contains an interval of length $\ge k(l-1)/2l = ks$.

Lemma 3.2 *There exists a* k_0 *so that* $A(n, [ks])$ *has prime divisors greater than* $(l+1)k$ *for all* $k \ge k_0$, $k \le \delta n$.

Proof Set $[ks] = t$ and consider the binomial coefficient

$$\binom{n+t}{t} = \frac{A(n, t)}{t!}$$

Every prime power q^α that divides a binormal coefficient $\binom{n+t}{t}$ satisfies $q^\alpha \le n + t$. Thus the hypothesis that all prime divisors are $< (l+1)k$ yields

$$\binom{n+t}{t} \le (n+t)^{\pi((l+1)k)} < (n+t)^{c(l+1)k/\log k} \tag{3.5}$$

for a suitable constant c. On the other hand

$$\binom{n+t}{t} \geq \left(\frac{n+t}{t}\right)^t. \tag{3.6}$$

Now set $(n+t)/t = C > 1/\delta$ and compare (3.5) and (3.6) to get

$$C^t < C^{c(l+1)k/\log k}(sk)^{c(l+1)k/\log k} \tag{3.7}$$

which is false for $k > k_0$ provided δ is small enough.

Theorem 3.3 *For each $\Delta > 1$ there exists only a finite number of integers k, n, m such that $k > 1, n + k \leq m \leq \Delta n$ and $A(m, k) \equiv 0 \pmod{A(n, k)}$ where the interval $[n + 1, n + k]$ contains a prime.*

Proof We first pick δ in Theorem 2.1 sufficiently small and then can restrict attention to a fixed integer l, $2 \leq l \leq \Delta + \delta$. By Lemma 3.2 we have $k < k_0$. Now pick one of the integers $x \in [n + 1, n + k]$ so that $lx \notin [m + 1, m + k]$. Then, by the same argument that we used in the proof of Lemma 3.1 we have $(x, y) < (l + 1)k$ for every $y \in [m + 1, m + k]$ and hence, if $x \mid A(m, k)$ we must have $n < x < ((l + 1)k)^k < ((l + 1)k_0)^{k_0}$.

We have not carried out the detailed estimates needed to show, for example, that the example stated at the beginning of this section is the unique solution for $\Delta = 2$, except for $A(4, 2)$ and $A(8, 2)$, but it would not be difficult to do so.

4. Open Questions

4.1. In view of Lemma 2.2, it would be interesting to know the smallest $m > 2k$ so that every prime in the interval $[k + 1, 2k]$ divides $A(m, k)$. In particular, is it true that $m \gg k^c$ for every c?

4.2. We know of no example with $n > 16$, $k > 2$, where $A(n, k)$ divides $A(m, k)$ and $n + k \leq m \leq 2n$. It would be interesting to find a bound for such n without the hypothesis that there exists a prime in the interval $[n + 1, n + k]$.

4.3. A question related to those discussed in this paper is to find solutions for $A(n, k) \mid A(n + k, n + 2k)$. Charles Grinstead has found the following examples:

n:	2	3	4	5	6	7	8	9
k:	4	3	206	1886	3472	3471	8170	8169

AMS (MOS) 1970 subject classification: 10A05.