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Note

On a Problem in Extremal Graph Theory

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The number $T^*(n, k)$ is the least positive integer such that every graph with $n = \binom{k+1}{2} + t$ vertices ($t \geqslant 0$) and at least $T^*(n, k)$ edges contains k mutually vertex-disjoint complete subgraphs $S_1, S_2, \ldots, S_k$ where S_i has i vertices, $1 \leqslant i \leqslant k$. Obviously $T^*(n, k) \geqslant T(n, k)$, the Turán number of edges for a K_k. It is shown that if $n \geqslant \frac{9}{8}k^2$ then equality holds and that there is $\epsilon > 0$ such that for $\binom{k+1}{2} \leqslant n \leqslant \binom{k+1}{2} + \epsilon k^2$ inequality holds. Further $T^*(n, k)$ is evaluated when $k > k_0(t)$.

INTRODUCTION

Let $G(n, m)$ denote a graph (V, E) with n vertices and m edges, K_i a complete graph with i vertices, and $\langle V' \rangle$ the subgraph of (V, E) induced by $V' \subseteq V$. The degree of $x \in V$ is denoted by $d(x)$, the minimum degree of the graph G by $\delta(G)$ and, for $V' \subseteq V$, $\varphi(V')$ denotes the number of edges with one end vertex in V' and the other in $V \backslash V'$.

In [1] Turán proved that every $G(n, T(n, k))$ contains a K_k, where

$$T(n, k) = \frac{k-2}{2(k-1)}(n^2 - r^2) + \binom{r}{2} + 1,$$

$r \equiv n \pmod{k-1}$ and $0 \leqslant r \leqslant k-2$. Furthermore he showed that the only $G(n, T(n, k) - 1)$ not containing a K_k has its vertex set partitioned into $k-1$ subsets of as near equal size as possible with any two vertices in different subsets adjacent. Dirac [2] and Erdös [3], among others, have extended this result.

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The Problem

For k a positive integer a graph $G(n, m)$ is said to possess property $P(k)$ if $n \geqslant \binom{k+1}{2}$ and it contains vertex-disjoint subgraphs $K_1, K_2, \ldots, K_k$. Find the least positive integer $T^*(n, k)$ such that every $G(n, T^*(n, k))$ has $P(k)$. Clearly $T^*(n, k) \geqslant T(n, k)$.

Results

Theorem 1. *If $n \geqslant \frac{9}{8}k^2$ then $T^*(n, k) = T(n, k)$.*

Theorem 2. *There exists $\epsilon > 0$ and $k_0 = k_0(\epsilon)$ such that if $k > k_0$ and $\binom{k+1}{2} \leqslant n \leqslant \binom{k+1}{2} + \epsilon k^2$ then $T^*(n, k) > T(n, k)$.*

Put $n = \binom{k+1}{2} + t$ and let $e(t, k)$ denote the number of edges of the n-vertex graph $X(t, k)$ whose complement consists of a K_{k+t+1} together with $n - k - t - 1$ isolated vertices.

Theorem 3. *There exists $k_0 = k_0(t)$ such that if $k > k_0$ then $T^*(n, k) = e(t, k) + 1$ and the only $G(n, e(t, k))$ without $P(k)$ is $X(t, k)$.*

Proof of Theorems

Proof of Theorem 1. The proof is by induction on k. Let $G = (V, E)$ be any $G(n, \geqslant T(n, k))$ and $n \geqslant \frac{9}{8}k^2$. Choose a subgraph $K_k = (V', E')$ of G with the property that $\varphi(V')$ is minimal and put $V^* = V \backslash V'$. If $\varphi(V') \leqslant k(n-k) - \binom{k}{2}$ then by using the induction hypothesis it can be shown that $\langle V^* \rangle$ has $P(k-1)$ and so G has $P(k)$. If, on the other hand, $\varphi(V') > k(n-k) - \binom{k}{2}$ put $A = \{x \in V^* \mid \{x, y\} \in E \text{ for all } y \in V'\}$, $B = V^* \backslash A$, whence $|B| < \binom{k}{2}$ and $|A| > \frac{5}{8}k^2 - \frac{1}{2}k$. Let p be a vertex of V' with maximal degree and $a \in A$. Then $\langle (V' \backslash \{p\}) \cup \{a\} \rangle = K_k$ and from the minimality of $\varphi(V')$ it follows that the number of edges $d_{V^*}(a)$ joining a to vertices of V^* satisfies $d_{V^*}(a) \geqslant d_{V^*}(p) - 1 \geqslant (1/k)\,\varphi(V') - 1 \geqslant |V^*| - (k/2)$, whence $\delta(\langle A \rangle) \geqslant |A| - [k/2]$. It is easy to show that any graph H with q vertices satisfying $\delta(H) \geqslant q - r$ has $P(k-1)$ if $q \geqslant \binom{k}{2} + \binom{r-1}{2}$ by picking a K_{k-1}, a K_{k-2} disjoint from it, a K_{k-3} disjoint from both of them etc. Putting $H = \langle A \rangle$, $r = [k/2]$ gives the result.

Proof of Theorem 2. The graph $G = (V, E)$ defined below has $n = \binom{k+1}{2} + t$ vertices, where $0 \leqslant t \leqslant \epsilon k^2$. Put $q = [k/10]$, $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \varnothing$. The subgraph $\langle V_1 \rangle$ is complete with $\frac{1}{2}(k-q)(k-q+3)$ vertices, $\langle V_2 \rangle$ has the maximum number of edges without containing a K_q (and is determined by Turán's theorem) and $\{x, y\} \in E$ whenever $x \in V_1$ and $y \in V_2$. It is left to the reader to verify that G does not have $P(k)$ and that $|E| > T(n, k)$ if ϵ is sufficiently small and k is sufficiently large.

LEMMA. *Let $k \geqslant 5$ and $G = (V, E)$ a graph without $P(k)$ having $n = \binom{k+1}{2} + t$ vertices, where $0 \leqslant t < k$. If G contains K_k then it is possible to find $Y \subseteq V$ such that $\langle Y \rangle = K_k$ and $\varphi(Y) \leqslant k(n-k) - (k+t+1)$ unless $G = X(t, k)$.*

Proof of Lemma. Suppose the Lemma is false. Choose $Y \subseteq V$ such that $\langle Y \rangle = K_k$ and $\varphi(Y) \geqslant k(n-k) - (k+t)$ minimal. Put $Z = V \backslash Y$, $A = \{z \in Z \mid \{y, z\} \in E$ for all $y \in Y\}$, $B = Z \backslash A$, whence $|B| \leqslant t + k$ and $|A| \geqslant \binom{k-1}{2} - 1$. Let $p \in Y$ satisfy $d(p) \geqslant d(q)$ for all $q \in Y$ and let $a \in A$. Then $\langle (Y \backslash \{p\}) \cup \{a\} \rangle = K_k$ and, by the minimality of $\varphi(Y)$, $d_z(a) \geqslant d_z(p) - 1 \geqslant (1/k)\, \varphi(Y) - 1 > |Z| - 3$. Thus $\delta(\langle A \rangle) \geqslant |A| - 2$. If $|A| \geqslant \binom{k-1}{2}$ then $\langle A \rangle$ has $P(k-2)$ and, for $1 \leqslant i \leqslant k-2$, it is possible to adjoin a vertex $z_i \in Z$ to the subgraph K_i of $\langle A \rangle$ to produce disjoint subgraphs $K_2, K_3, \ldots, K_{k-1}$ of $\langle Z \rangle$, whence G has $P(k)$. Thus $|A| = \binom{k-1}{2} - 1$, $|B| = t + k$, $\varphi(Y) = k(n-k) - (k+t)$ and each $b \in B$ satisfies $d_Y(b) = k - 1$. Vertex-disjoint subgraphs $K_2, K_3, \ldots, K_{k-2}$ of $\langle A \rangle$ may be shown to exist. If there are k independent edges $\{p_i, q_i\}$ where $p_i \in Y$, $q_i \in B$, and $1 \leqslant i \leqslant k$ then it is possible to adjoint one of these, say $\{p_{k-2}, q_{k-2}\}$, to K_{k-2} to give a K_k, another, say $\{p_{k-3}, q_{k-3}\}$, to K_{k-3} to give a $K_{k-1}, \ldots$, showing that G has $P(k)$. Otherwise there is $y \in Y$ not adjacent to any vertex of B. Let $x \in Y \backslash \{y\}$ and $a \in A$. Then x is adjacent to every vertex of Z, $\langle (Y \backslash \{x\}) \cup \{a\} \rangle = K_k$ and, by the minimality of $\varphi(Y)$, a is adjacent to every other vertex of Z. If $\langle B \rangle$ contains an edge $\{b_1, b_2\}$ then, for $2 \leqslant i \leqslant k-2$, adjoining $b_{i+1} \in B \backslash \{b_1, b_2\}$ to K_i shows that G has $P(k)$. If $\langle B \rangle$ contains no edge then $G = X(t, k)$.

Proof of Theorem 3. Let $G = (V, E)$ be a $G(n, \geqslant e(t, k))$ and

$$k > k_0 = \binom{\binom{t+20}{2} + t}{2} + t + 19.$$

By elementary algebra it can be shown that $e(t, k) \geqslant T(n, k)$ when $k > t$ and $k \geqslant 20$, so that G contains K_k. By the Lemma if $G \neq X(t, k)$ and G does not have $P(k)$ then $\langle Z \rangle$ is a $G(\binom{k}{2} + t, \geqslant e(t, k-1) + 1)$. Applying the same argument to $\langle Z \rangle$ yields a subgraph $\langle Z' \rangle$ which is a $G(\binom{k-1}{2} + t, \geqslant e(t, k-2) + 2)$. Repeated application yields a $G(\binom{t+20}{2} + t, \geqslant e(t, t+19) + k - t - 19)$ which has more edges than the complete graph, a contradiction.

AN UNSOLVED PROBLEM

In view of Theorems 1 and 2 the following problem remains to be solved. Evaluate $f(k)$, the smallest integer such that whenever $n \geqslant f(k)$ then $T^*(n, k) = T(n, k)$. In terms of $f(k)$ Theorem 1 becomes $f(k) \leqslant \frac{9}{8}k^2$.

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