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Partitions of the Natural Numbers into Infinitely Oscillating Bases and Nonbases

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Abstract. The set A of nonnegative integers is a basis if every sufficiently large integer x can be written in the form $x=a+a'$ with $a, a' \in A$. If A is not a basis, then it is a nonbasis. We construct a partition of the natural numbers into a basis A and a nonbasis B such that, as random elements are moved one at a time from A to B, from B to A, from A to $B, \ldots$, the set A oscillates from basis to nonbasis to basis ... and the set B oscillates simultaneously from nonbasis to basis to nonbasis....

1. Introduction

Let A be an infinite subset of the natural numbers $\mathbb{N}=\{0, 1, 2, \ldots\}$. Then A is an asymptotic basis of order 2, or, simply, a *basis*, if every sufficiently large number can be written in the form a_i+a_j, where $a_i, a_j \in A$. If the set A is not a basis, then it is called an asymptotic nonbasis of order 2, or, simply, a *nonbasis*.

The set A is a *minimal basis* if A is a basis, but, for any $a \in A$, the set $A\setminus\{a\}$ is a nonbasis. Similarly, the set A is a *maximal nonbasis* if A is a nonbasis, but, for any natural number $b \notin A$, the set $A\cup\{b\}$ is a basis. Minimal bases and maximal nonbases were introduced by Stöhr [5] and Nathanson [4], and studied further by Härtter [3] and Erdös and Nathanson [1, 2].

Minimal bases and maximal nonbases are examples of sets which oscillate once from basis to nonbasis or from nonbasis to basis by the deletion from or addition to the set of a single element. There also exist sets which exhibit two oscillations. Erdös and Nathanson [2] have constructed a basis A such that, for any $a \in A$, the set $A\setminus\{a\}$ is a nonbasis, and, for any $b \notin A\setminus\{a\}$, the set $(A\setminus\{a\})\cup\{b\}$ is again a basis. They also constructed a nonbasis A such that, for any $b \notin A$, the set $A\cup\{b\}$ is a basis, and, for any $a \in A\cup\{b\}$, the set $(A\cup\{b\})\setminus\{a\}$ is again a nonbasis. But no example had been constructed of a set which would oscillate infinitely often from basis to nonbasis to basis to nonbasis ... by successive deletions from and additions to the set of single elements. Such a set can be precisely described in the following way. Let A be an infinite set of natural numbers, and let S and

T be finite sets such that $S \subset A$ and $T \subset \mathbb{N} \setminus A$. Then A is an *infinitely oscillating basis* if $(A \setminus S) \cup T$ is a basis if and only if $|S| \leq |T|$. Similarly, let B be an infinite set of natural numbers, and let S and T be finite sets such that $T \subset B$ and $S \subset \mathbb{N} \setminus B$. Then B is an *infinitely oscillating nonbasis* if $(B \cup S) \setminus T$ is a nonbasis if and only if $|S| \leq |T|$. Clearly, if A is an infinitely oscillating basis, then $A \setminus \{a\}$ is an infinitely oscillating nonbasis for any $a \in A$. Similarly, if B is an infinitely oscillating nonbasis, then $B \cup \{a\}$ is an infinitely oscillating basis for any $a \notin B$.

Nathanson [4] asked if there existed a partition of the natural numbers into a minimal basis A and a maximal nonbasis B. This partition would have the property that A is a basis and B is a nonbasis, but, if any element $a \in A$ is moved to B, then $A \setminus \{a\}$ becomes a nonbasis and $B \cup \{a\}$ becomes a basis. One can ask, further, for such a partition with the additional property that if any element $b \in B \cup \{a\}$ is moved to $A \setminus \{a\}$, then $(B \cup \{a\}) \setminus \{b\}$ becomes a nonbasis and $(A \setminus \{a\}) \cup \{b\}$ becomes a basis again. Indeed, one could wish for a partition of $\mathbb{N}$ into a basis A and a nonbasis B such that, as random elements are moved one at a time from one set of the partition to the other, the set which is a basis becomes a nonbasis and the set which is a nonbasis becomes a basis. This is equivalent to requiring a partition of the natural numbers into two sets, one of which is an infinitely oscillating basis and the other an infinitely oscillating nonbasis. The purpose of this paper is to construct such a partition. In particular, this proves the existence of infinitely oscillating bases.

THEOREM. *There exists a partition of the natural numbers $\mathbb{N}$ into two disjoint sets A and B such that A is an infinitely oscillating basis and B is an infinitely oscillating nonbasis.*

2. A Critical Lemma

The following notation will be used consistently in this paper. If A is a set of numbers, then the sumset $2A = \{a + a' \mid a, a' \in A\}$. By $[M, N]$ we denote the interval of integers $x = M, M+1, \ldots, N$. Let $N_k > 2N_{k-1}$, where $N_k = 2n_k + 1$ and $n_k = 2m_k$ is even. The interval $[N_{k-1}+1, N_k]$ will be divided into the following three subintervals:

$$I'_k = [N_{k-1}+1, n_k], \qquad I''_k = [n_k+1, N_k - N_{k-1} - 1], \qquad I'''_k = [N_k - N_{k-1}, N_k].$$

By A'_k and B'_k (resp. A''_k and B''_k, A'''_k and B'''_k) we denote subsets of I'_k

(resp. I''_k, I'''_k) which partition the interval I'_k (resp. I''_k, I'''_k). Let

$$I_k = I'_k \cup I''_k = [N_{k-1}+1, N_k - N_{k-1} - 1]$$

and let $A_k = A'_k \cup A''_k$ and $B_k = B'_k \cup B''_k$. Then the sets A_k and B_k partition the interval I_k and the sets $A_k \cup A'''_k$ and $B_k \cup B'''_k$ partition the interval $[N_{k-1}+1, N_k]$.

The cardinality of the finite set A is denoted $|A|$.

LEMMA 1. *Let $x \in [2P+2, P+Q+1]$. Then the number of subsets A of $[P+1, Q]$ such that $x \notin 2A$ is less than*

$$\left(\frac{\sqrt{3}}{2}\right)^{x-2P} 2^{Q-P+1}.$$

Proof. Let $A \subseteq [P+1, Q]$ with $x \notin 2A$. Suppose $x = 2x'+1$ is odd. We divide $[P+1, Q]$ into the interval $[x-P, Q]$ and the $x'-P$ pairs $\{r, x-r\}$ with $r = P+1, P+2, \ldots, x'$. Then A can contain any of the $2^{Q-(x-P)+1}$ subsets of $[x-P, Q]$. On the other hand, A can contain at most one element from each pair $\{r, x-r\}$, and so there are three choices for the distribution of each pair $\{r, x-r\}$ in A (either $r \in A$, $x-r \notin A$, or $r \notin A$, $x-r \in A$, or $r \notin A$, $x-r \notin A$). Therefore, the number of ways to choose A is exactly

$$3^{x'-P} 2^{Q-(x-P)+1} = 3^{(x-2P-1)/2} 2^{Q-P+1-(x-2P)} < \left(\frac{\sqrt{3}}{2}\right)^{x-2P} 2^{Q-P+1}.$$

Similarly, if $x = 2x'$ is even, we divide $[P+1, Q]$ into the interval $[x-P, Q]$, the singleton $\{x'\}$, and the $x'-P-1$ pairs $\{r, x-r\}$, where $r = P+1, P+2, \ldots, x'-1$. Clearly, $x' \notin A$, and the number of ways to choose A is exactly

$$3^{x'-P-1} 2^{Q-(x-P)+1} = 3^{(x-2P-2)/2} 2^{Q-P+1-(x-2P)} < \left(\frac{\sqrt{3}}{2}\right)^{x-2P} 2^{Q-P+1}.$$

LEMMA 2. *Let $x \in [P+Q+1, 2Q]$. Then the number of subsets A of $[P+1, Q]$ such that $x \notin 2A$ is less than*

$$\left(\frac{\sqrt{3}}{2}\right)^{2Q-x} 2^{Q-P}.$$

Proof. Let $A\subset[P+1,Q]$ with $x\notin 2A$. Suppose $x=2x'+1$ is odd. We divide $[P+1,Q]$ into the interval $[P+1, x-Q-1]$ and the $Q-x'$ pairs $\{x-r,r\}$ where $r=x'+1, x'+2,\ldots,Q$. Then the number of ways to choose A is exactly

$$3^{Q-x'}2^{x-Q-1-P}=3^{(2Q-x+1)/2}2^{Q-P-1-(2Q-x)}<\left(\frac{\sqrt{3}}{2}\right)^{2Q-x}2^{Q-P}.$$

Similarly, if $x=2x'$ is even, we divide $[P+1,Q]$ into the interval $[P+1,x-Q-1]$, the singleton $\{x'\}$, and the $Q-x'$ pairs $\{x-r,r\}$, where $r=x'+1,x'+2,\ldots,Q$. Then the number of ways to choose A is exactly

$$3^{Q-x'}2^{x-Q-1-P}=3^{(2Q-x)/2}2^{Q-P-1-(2Q-x)}<\left(\frac{\sqrt{3}}{2}\right)^{2Q-x}2^{Q-P}.$$

LEMMA 3. *Let $d\geq 1$. Then the number of subsets A of $[P+1,Q]$ such that*

$$a\in A \quad \text{and} \quad a\leq Q-d \quad \text{implies} \quad a+d\in A \tag{*}$$

does not exceed

$$\left(\frac{Q-P}{d}+2\right)^d.$$

Similarly, the number of subsets A of $[P+1,Q]$ such that

$$a\in A \quad \text{and} \quad a\geq P+1+d \quad \text{implies} \quad a-d\in A \tag{**}$$

does not exceed

$$\left(\frac{Q-P}{d}+2\right)^d.$$

Proof. The interval $[P+1,Q]$ can be partitioned into d disjoint arithmetic progressions with difference d, each of length at most $(Q-P)/d+1$. Suppose that $A\subset[P+1,Q]$ satisfies (*) (resp. (**)). Then A is the disjoint union of terminal (resp. initial) segments of the d arithmetic progressions, and each of these segments is determined by its initial (resp. terminal) element, which can be chosen in at most $(Q-P)/d+2$ ways. Since there are d progressions, the number of $A \subset [P+1, Q]$ which satisfy (*) (resp. (**)) is at most $((Q-P)/d+2)^d$.

LEMMA 4. *There exists a constant c such that, given a nonnegative integer* N_{k-1}, *then for all sufficiently large* $N_k = 2n_k + 1$ *there is a partition of the interval* $I_k = [N_{k-1}+1, N_k - N_{k-1} - 1]$ *into two sets* A_k *and* B_k *such that*

(i) $N_k \notin 2A_k \cup 2B_k$

(ii) $[N_k + 1, 2N_k - 2N_{k-1} - 2 - c] \subset 2A_k \cap 2B_k$.

Furthermore, if N_{k-1} *is sufficiently greater than* N_{k-2}, *and if there is a partition of the interval* $I_{k-1} = [N_{k-2}+1,\ N_{k-1} - N_{k-2} - 1]$ *into two sets* A_{k-1} *and* B_{k-1} *such that*

(iii) $N_{k-1} \notin 2A_{k-1} \cup 2B_{k-1}$

(iv) $[N_{k-1}+1, 2N_{k-1} - 2N_{k-2} - 2 - c] \subset 2A_{k-1} \cap 2B_{k-1}$

then there is a partition of I_k *into sets* A_k *and* B_k *which satisfy* (i), (ii), *and also*

(v) $[N_{k-1}+1, N_{k-1}] \subset 2(A_k \cup A_{k-1}) \cap 2(B_k \cup B_{k-1})$.

Proof. Let us call a partition $I_k = A_k \cup B_k$ *permissible* if $N_k \notin 2A_k \cup 2B_k$. Since I_k is symmetric with respect to $N_k/2$, then $x \in A_k$ if and only if $N_k - x \in B_k$. Let $I'_k = [N_{k-1}+1, n_k]$ and $I''_k = [n_k + 1, N_k - N_{k-1} - 1]$. Let $A'_k = A_k \cap I'_k$, $A''_k = A_k \cap I''_k$, $B'_k = B_k \cap I'_k$, and $B''_k = B_k \cap I''_k$. Then $x \in A'_k$ if and only if $N_k - x \in B''_k$, and $x \in B'_k$ if and only if $N_k - x \in A''_k$. Clearly, if $I_k = A_k \cup B_k$ is a permissible partition, then each one of the four sets A'_k, A''_k, B'_k, B''_k uniquely determines the other three. Since A'_k can be any subset of $I'_k = [N_{k-1}+1, n_k]$, it follows that there are exactly $2^{n_k - N_k - 1}$ permissible partitions of I_k. We shall prove that for any $\varepsilon > 0$ there exists a constant c such that, for all sufficiently large N_k, the number of permissible partitions of I_k which also satisfy condition (ii) is greater than $(1-\varepsilon)2^{n_k - N_{k-1}}$. Moreover, for this constant c, if N_{k-1} is sufficiently greater than N_{k-2} and if there exists a partition $I_{k-1} = A_{k-1} \cup B_{k-1}$ which satisfies conditions (iii) and (iv), then the number of permissible partitions of I_k which satisfy both conditions (ii) and (v) is greater than $(1-\varepsilon)2^{n_k - N_{k-1}}$.

Let $\varepsilon > 0$, let $\varepsilon' = \varepsilon/18$, and choose the constant $c \geq 2$ so that

$$\sum_{t=c}^{\infty} \left(\frac{\sqrt{3}}{2}\right)^t < \varepsilon'.$$

Let $N_k=2n_k+1$, where $n_k=2m_k$ and $m_k\geq 2N_{k-1}+c+1$ and also

$$\sum_{d=1}^{c}(n_k+2)^d<\varepsilon' 2^{n_k-n_{k-1}}.$$

The proof is in seven steps.

I. Let $x\in[N_k+c-1,\,n_k+N_k-N_{k-1}]$. By Lemma 1, the number of subsets A_k'' of $I_k''=[n_k+1,\,N_k-N_{k-1}-1]$ such that $x\notin 2A_k''$ is less than

$$\left(\frac{\sqrt{3}}{2}\right)^{x-2n_k}2^{n_k-N_{k-1}+1}.$$

Therefore, the number of $A_k''\subset I_k''$ such that $x\notin 2A_k''$ for some $x\in[N_k+c-1,\,n_k+N_k-N_{k-1}]$ is less than

$$\sum_{x=N_k+c-1}^{n_k+N_k-N_{k-1}}\left(\frac{\sqrt{3}}{2}\right)^{x-2n_k}2^{n_k-N_{k-1}+1}=2^{n_k-N_{k-1}+1}\sum_{t=c}^{N_k-N_{k-1}-n}\left(\frac{\sqrt{3}}{2}\right)^t<2\varepsilon' 2^{n_k-N_{k-1}}.$$

Since each set $A_k''\subset I_k''$ completely determines a permissible partition $I_k=A_k\cup B_k$, we conclude that the number of permissible partitions with $x\notin 2A_k$ for some $x\in[N_k+c-1,\,n_k+N_k-N_{k-1}]$ is less than $2\varepsilon' 2^{n_k-N_{k-1}}$.

II. Let $x\in[n_k+N_k-N_{k-1},\,2N_k-2N_{k-1}-2-c]$. By Lemma 2, the number of $A_k''\subset I_k''$ such that $x\notin 2A_k''$ is less than

$$\left(\frac{\sqrt{3}}{2}\right)^{2N_k-2N_{k-1}-2-x}2^{n_k-N_{k-1}}.$$

Therefore, the number of $A_k''\subset I_k''$ such that $x\notin 2A_k''$ for some $x\in[n_k+N_k-N_{k-1},\,2N_k-2N_{k-1}-2-c]$ is less than

$$\sum_{x=n_k+N_k-N_{k-1}}^{2N_k-2N_{k-1}-2-c}\left(\frac{\sqrt{3}}{2}\right)^{2N_k-2N_{k-1}-2-x}2^{n_k-N_{k-1}}=2^{n_k-N_{k-1}}\sum_{t=c}^{N_k-N_{k-1}-n_k-2}\left(\frac{\sqrt{3}}{2}\right)^t<\varepsilon' 2^{n_k-N_{k-1}}.$$

It follows that the number of permissible partitions $I_k=A_k\cup B_k$ such that $x\notin 2A_k$ for some $x\in[n_k+N_k-N_{k-1},\,2N_k-2N_{k-1}-2-c]$ is less than $\varepsilon' 2^{n_k-N_{k-1}}$.

III. Let $x\in[N_k+1,\,N_k+c-2]$. Then $x=N_k+d$ for some $d\in[1,c-2]$. Let $I_k=A_k\cup B_k$ be a permissible partition such that $x\notin 2A_k$. Let $A_k=A_k'\cup A_k''$, and let $a\in A_k''$ with $a\geq n_k+1+d$. Then $x-a\in I_k'=A_k'\cup B_k'$. But $a\in A_k''$ and

$x \notin 2A_k$ imply $x-a \notin A'_k$. Therefore, $x-a \in B'_k$. Since $I_k = A_k \cup B_k$ is a permissible partition, $N_k-(x-a)=a-d \in A''_k$. That is, $A''_k \subset [n_k+1, N_k-N_{k-1}-1]$, and if $a \in A''_k$ and $a \geq n_k+1+d$, then $a-d \in A''_k$. By Lemma 3, the number of such sets A''_k does not exceed

$$\left(\frac{n_k-N_{k-1}}{d}+2\right)^d < (n_k+2)^d.$$

Therefore, the number of permissible partitions $I_k = A_k \cup B_k$ such that $x \notin 2A_k$ for some $x \in [N_k+1, N_k+c-2]$ is less than

$$\sum_{d=1}^{c-2} (n_k+2)^d < \varepsilon' 2^{n_k-N_{k-1}}.$$

Combining the results of I–III, we conclude that the number of permissible partitions $I_k = A_k \cup B_k$ such that $x \notin 2A_k$ for some $x \in [N_k+1, 2N_k-2N_{k-1}-2-c]$ is less than $4\varepsilon' 2^{n_k-N_{k-1}}$. Similarly, the number of permissible partitions $I_k = A_k \cup B_k$ such that $x \notin 2B_k$ for some $x \in [N_k+1, 2N_k-2N_{k-1}-2-c]$ is less than $4\varepsilon' 2^{n_k-N_{k-1}}$. Therefore, condition (ii) fails to hold for less than $8\varepsilon' 2^{n_k-N_{k-1}} < \varepsilon 2^{n_k-N_{k-1}}$ permissible partitions of I_k. This proves the first part of Lemma 4.

IV. Let $x \in [2N_{k-1}+c, n_k+N_{k-1}+1]$. By Lemma 1, the number of subsets A'_k of $I'_k = [N_{k-1}+1, n_k]$ such that $x \notin 2A'_k$ is less than

$$\left(\frac{\sqrt{3}}{2}\right)^{x-2N_{k-1}} 2^{n_k-N_{k-1}+1}.$$

Therefore, the number of $A'_k \subset I'_k$ such that $x \notin 2A'_k$ for some $x \in [2N_{k-1}+c, n_k+N_{k-1}+1]$ is less than

$$\sum_{x=2N_{k-1}+c}^{n_k+N_{k-1}+1} \left(\frac{\sqrt{3}}{2}\right)^{x-2N_{k-1}} 2^{n_k-N_{k-1}+1} = 2^{n_k-N_{k-1}+1} \sum_{t=c}^{n_k-N_{k-1}+1} \left(\frac{\sqrt{3}}{2}\right)^t < 2\varepsilon' 2^{n_k-N_{k-1}}.$$

Then the number of permissible partitions $I_k = A_k \cup B_k$ such that $x \notin 2A_k$ for some $x \in [2N_{k-1}+c, n_k+N_{k-1}+1]$ is less than $2\varepsilon' 2^{n_k-N_{k-1}}$.

V. Let $x \in [n_k+N_{k-1}+1, N_k-c-1]$. By Lemma 2, the number of $A'_k \subset I'_k$ such that $x \notin 2A'_k$ is less than

$$\left(\frac{\sqrt{3}}{2}\right)^{2n_k-x} 2^{n_k-N_{k-1}} = \left(\frac{\sqrt{3}}{2}\right)^{N_k-1-x} 2^{n_k-N_{k-1}}.$$

Therefore, the number of $A_k' \subset I_k'$ such that $x \notin 2A_k'$ for some $x \in [n_k + N_{k-1} + 1, N_k - c - 1]$ is less than

$$\sum_{x=n_k+N_{k-1}+1}^{N_k-c-1} \left(\frac{\sqrt{3}}{2}\right)^{N_k-1-x} 2^{n_k-N_{k-1}} = 2^{n_k-N_{k-1}} \sum_{t=c}^{n_k-N_{k-1}-1} \left(\frac{\sqrt{3}}{2}\right)^{t} < \varepsilon' 2^{n_k-N_{k-1}}.$$

Therefore, the number of permissible partitions $I_k = A_k \cup B_k$ such that $x \notin 2A_k$ for some $x \in [n_k + N_{k-1} + 1, N_k - c - 1]$ is less than $\varepsilon' 2^{n_k - N_{k-1}}$.

VI. Let $x \in [N_k - c, N_k - 1]$. Then $x = N_k - d$ for some $d \in [1, c]$. Let $I_k = A_k \cup B_k$ be a permissible partition such that $x \notin 2A_k$. Let $A_k = A_k' \cup A_k''$, and let $a \in A_k'$ with $a \leq n_k - d$. Then $x - a \in I_k'' = A_k'' \cup B_k''$. But $a \in A_k'$ and $x \notin 2A_k$ imply $x - a \notin A_k''$. Therefore, $x - a \in B_k''$. Since $I_k = A_k \cup B_k$ is a permissible partition, $N_k - (x - a) = a + d \in A_k'$. That is, $A_k' \subset [N_{k-1} + 1, n_k]$, and if $a \in A_k'$ and $a \leq n_k - d$, then $a + d \in A_k'$. By Lemma 3, the number of such sets A_k' does not exceed

$$\left(\frac{n_k - N_{k-1}}{d} + 2\right)^d < (n_k + 2)^d.$$

Therefore, the number of permissible partitions $I_k = A_k \cup B_k$ such that $x \notin 2A_k$ for some $x \in [N_k - c, N_k - 1]$ is less than

$$\sum_{d=1}^{c} (n_k + 2)^d < \varepsilon' 2^{n_k - N_{k-1}}.$$

VII. Let $x \in [2N_{k-1} - 2N_{k-2} - 1 - c, 2N_{k-1} + c - 1]$. Now we suppose that there is a partition of the interval $I_{k-1} = [N_{k-2} + 1, N_{k-1} - N_{k-2} - 1]$ into two sets A_{k-1} and B_{k-1} that satisfy conditions (iii) and (iv), and that $N_{k-1} = 2n_{k-1} + 1$, where $n_{k-1} = 2m_{k-1}$ is even, and $m_{k-1} \geq 2N_{k-2} + c + 1$, and

$$\frac{2N_{k-2} + 2c + 1}{2^{m_{k-1}}} < \varepsilon'.$$

Then $J = [n_{k-1} - m_{k-1} + 1, n_{k-1} + m_{k-1}] = [m_{k-1} + 1, 3m_{k-1}] \subset I_{k-1}$, and J is symmetric with respect to $N_{k-1}/2$. By condition (iii) we have $N_{k-1} \notin 2A_{k-1} \cup 2B_{k-1}$, and so J contains exactly m_{k-1} elements of A_{k-1} and m_{k-1} elements of B_{k-1}. Moreover, if $a \in J$, then $x - a \in I_k'$, since $x - a \leq x \leq n_k$ and

$$x - a \geq (2N_{k-1} - 2N_{k-2} - 1 - c) - 3m_{k-1} = N_{k-1} - 2N_{k-2} - c + m_{k-1} \geq N_{k-1} + 1.$$

Let $I_k = A_k \cup B_k$ be a permissible partition such that $x \notin 2(A_k \cup A_{k-1})$. If a is one of the m_{k-1} elements of $J \cap A_{k-1}$, then $x - a \in I'_k$. But $x - a \notin A'_k$ since $x \notin 2(A_k \cup A_{k-1})$. Therefore, A'_k is a subset of a set with $n_k - N_{k-1} - m_{k-1}$ elements, and so A'_k can be chosen in at most $2^{n_k - N_{k-1} - m_{k-1}}$ ways. Therefore, the number of permissible partitions $I_k = A_k \cup B_k$ with $x \notin 2(A_k \cup A_{k-1})$ is at most $2^{n_k - N_{k-1} - m_{k-1}}$, and the number of permissible partitions $I_k = A_k \cup B_k$ with $x \notin 2(A_k \cup A_{k-1})$ for some $x \in [2N_{k-1} - 2N_{k-2} - 1 - c, 2N_{k-1} + c - 1]$ is at most

$$(2N_{k-2} + 2c + 1)2^{n_k - N_{k-1} - m_{k-1}} = \frac{2N_{k-2} + 2c + 1}{2^{m_{k-1}}} 2^{n_k - N_{k-1}} < \varepsilon' 2^{n_k - N_{k-1}}.$$

Combining the results of IV–VII, we conclude that the number of permissible partitions $I_k = A_k \cup B_k$ such that $x \notin 2(A_k \cup A_{k-1})$ for some $x \in [2N_{k-1} - 2N_{k-2} - 1 - c, N_k - 1]$ is less than $5\varepsilon' 2^{n_k - N_{k-1}}$. Similarly, the number of permissible partitions $I_k = A_k \cup B_k$ such that $x \notin 2(B_k \cup B_{k-1})$ for some $x \in [2N_{k-1} - 2N_{k-2} - 1 - c, N_k - 1]$ is less than $5\varepsilon' 2^{n_k - N_{k-1}}$. Combining this with condition (iv), we conclude that

$$[N_{k-1} + 1, N_k - 1] \subseteq 2(A_k \cup A_{k-1}) \cap 2(B_k \cup B_{k-1})$$

for all but at most $10\varepsilon' 2^{n_k - N_{k-1}}$ permissible partitions of I_k. Putting together the results of I–VII, we see that conditions (ii) and (v) fail to hold for less than $18\varepsilon' 2^{n_k - N_{k-1}} = \varepsilon 2^{n_k - N_{k-1}}$ permissible partitions $I_k = A_k \cup B_k$. This finishes the proof of Lemma 4.

CRITICAL LEMMA. *There exists an increasing sequence* $0 = N_0 < N_1 < N_2 < \cdots$ *and disjoint sets* A_k *and* B_k *with* $A_k \cup B_k = [N_{k-1} + 1, N_k - N_{k-1} - 1] = I_k$ *for all* $k \geq 1$ *such that, if* $A^* = \cup_{k=1}^{\infty} A_k$ *and* $B^* = \cup_{k=1}^{\infty} B_k$, *then*

(i) $N_k \notin 2A^* \cup 2B^*$ *for all* k, *and*

(ii) *If* F *is any finite set of integers, then*

$$x \in 2(A^* \setminus F) \cap 2(B^* \setminus F)$$

for all sufficiently large $x \neq N_k$.

Proof. By Lemma 4, there exists an integer $N_1 > 0$ and disjoint sets A_1 and B_1 with $[1, N_1 - 1] = A_1 \cup B_1$ such that $N_1 \notin 2A_1 \cup 2B_1$ and

$[N_1+1, 2N_1-2-c] \subset 2A_1 \cap 2B_1$. Again by Lemma 4 there exists $N_2 > N_1$ and disjoint sets A_2 and B_2 with $[N_1+1, N_2-N_1-1] = A_2 \cup B_2$ such that conditions (i), (ii), and (v) of Lemma 4 are satisfied for $k=2$. We proceed by induction to construct an infinite sequence of integers $0 = N_0 < N_1 < N_2 < \cdots$ and disjoint sets A_k and B_k such that $I_k = A_k \cup B_k$ and conditions (i), (ii), and (v) of Lemma 4 are satisfied. Now set $A^* = \bigcup_{k=1}^{\infty} A_k$ and $B^* = \bigcup_{k=1}^{\infty} B_k$. It follows from condition (i) of Lemma 4 and the shape of the intervals I_k that $N_k \notin 2A^* \cap 2B^*$ for all k.

Let F be any finite set of integers. Then $F \subset [0, N_p]$ for sufficiently large p. Let $x > N_{p+1}$ and $x \neq N_k$ for all k. Then $x \in [N_{k-1}+1, N_k-1]$ for some $k \geq p+2$, and so $x \in 2(A_k \cup A_{k-1}) \cap 2(B_k \cup B_{k-1})$. But $A_k \cup A_{k-1} \subset A^* \setminus F$ and $B_k \cup B_{k-1} \subset B^* \setminus F$ since $k-1 \geq p+1$, and so $x \in 2(A^* \setminus F) \cap 2(B^* \setminus F)$. This proves the Critical Lemma.

3. Proof of the Theorem

Let $0 = N_0 < N_1 < N_2 < \cdots$ be an increasing sequence of integers, and let A_k and B_k be a partition of the interval $I_k = [N_{k-1}+1, N_k - N_{k-1} - 1]$ such that $A^* = \bigcup_{k=1}^{\infty} A_k$ and $B^* = \bigcup_{k=1}^{\infty} B_k$ satisfy the conclusions of the Critical Lemma. We shall construct a partition of the natural numbers into an infinitely oscillating basis A and an infinitely oscillating nonbasis B with $A^* \subset A$ and $B^* \subset B$.

Set $I_k''' = [N_k - N_{k-1}, N_k]$ for $k \geq 1$. In particular, $I_1''' = [N_1, N_1] = \{N_1\}$. We shall construct partitions of the intervals I_k''' into disjoint sets A_k''' and B_k'''. Let $A_1''' = \{N_1\}$ and $B_1''' = \phi$. Suppose that partitions $I_j''' = A_j''' \cup B_j'''$ have been determined for all $j \leq k-1$. We construct A_k''' and B_k'''.

Let p be an integer such that

$$1 \leq p \leq 1 + \sum_{j=1}^{k-2} |A_j| = 1 + \sum_{j=1}^{k-2} |B_j|.$$

Suppose that k is even. Choose $S \subset \bigcup_{j=1}^{k-1} (A_j \cup A_j''') \cup \{0\}$ with $|S| = p$, and choose $T \subset \bigcup_{j=1}^{k-1} (B_j \cup B_j''')$ with $|T| = p-1$. Let $a \in \bigcup_{j=1}^{k-1} (A_j \cup A_j''') \cup \{0\}$. If $a \in S$, put $N_k - a \in A_k'''$. If $a \notin S$, put $N_k - a \in B_k'''$. Let $b \in \bigcup_{j=1}^{k-1} (B_j \cup B_j''')$. If $b \in T \cup B_{k-1}$, put $N_k - b \in B_k'''$. If $b \notin T \cup B_{k-1}$, put $N_k - b \in A_k'''$. Since the sets $\{0\}$, A_j, A_j''', B_j, B_j''' for $j = 1, 2, \ldots, k-1$ are disjoint and partition $[0, N_{k-1}]$, and since the numbers in I_k''' are precisely those of the form $N_k - x$ for $x \in [0, N_{k-1}]$, it follows that the sets A_k''' and B_k''' partition the interval I_k'''.

We can count the number of representations of N_k. Clearly, N_k has exactly $|S|=p$ representations of the form $N_k=a+a'$ with $a, a'\in \bigcup_{j=1}^{k}(A_j\cup A''_j)\cup\{0\}$, namely, those with $a\in S$ and $a'=N_k-a$. Also, N_k has exactly $|T\cup B_{k-1}|=p-1+n_{k-1}-N_{k-2}$ representations in the form $N_k=b+b'$ with $b, b'\in \bigcup_{j=1}^{k}(B_j\cup B''_j)$, namely, those with $b\in T\cup B_{k-1}$ and $b'=N_k-b$.

Now suppose that k is odd. Choose $T^{\#}\subset \bigcup_{j=1}^{k-1}(B_j\cup B''_j)\cup\{0\}$ with $|T^{\#}|=p$, and choose $S^{\#}\subset \bigcup_{j=1}^{k-2}(A_j\cup A''_j)$ with $|S^{\#}|=p-1$. Let $b\in \bigcup_{j=1}^{k-1}(B_j\cup B''_j)\cup\{0\}$. If $b\in T^{\#}$, put $N_k-b\in B''_k$. If $b\notin T^{\#}$, put $N_k-b\in A''_k$. Let $a\in \bigcup_{j=1}^{k-1}(A_j\cup A''_j)$. If $a\in S^{\#}\cup A_{k-1}$, put $N_k-a\in A''_k$. If $a\notin S^{\#}\cup A_{k-1}$, put $N_k-a\in B''_k$. This determines a partition $I''_k=A''_k\cup B''_k$ such that N_k has exactly $|T^{\#}|=p$ representations as a sum of two elements of $\bigcup_{j=1}^{k}(B_j\cup B''_j)\cup\{0\}$ and N_k has exactly $|S^{\#}\cup A_{k-1}|=p-1+n_{k-1}-N_{k-2}$ representations as a sum of two elements of $\bigcup_{j=1}^{k}(A_j\cup A''_j)$.

We can now partition the natural numbers into two disjoint sets A and B, where

$$A=\sum_{k=1}^{\infty}(A_k\cup A''_k)\cup\{0\}=A^*\cup\left(\bigcup_{k=1}^{\infty}A''_k\right)\cup\{0\}$$

$$B=\bigcup_{k=1}^{\infty}(B_k\cup B''_k)=B^*\cup\left(\bigcup_{k=1}^{\infty}B''_k\right).$$

The sets A''_k and B''_k are constructed inductively in such a way that, for every $p\geq 1$, every pair of sets S, T (where $S\subset A$ and $|S|=p$, and $T\subset B$ and $|T|=p-1$) is used to construct partitions $I''_k=A''_k\cup B''_k$ for infinitely many even integers k, and every pair of sets $T^{\#}$, $S^{\#}$ (where $T^{\#}\subset B\cup\{0\}$ and $|T^{\#}|=p$, and $S^{\#}\subset A\setminus\{0\}$ and $|S^{\#}|=p-1$) is used to construct partitions $I''_k=A''_k\cup B''_k$ for infinitely many odd integers k.

We shall prove that A is an infinitely oscillating basis. Let S be a finite subset of A, say, $|S|=p$. Since $A^*\subset A$, it follows from the Critical Lemma that all sufficiently large $x\neq N_k$ can be written in the form $x=a+a'$ with $a, a'\in A\setminus S$. If k is odd, then N_k has at least $|A_{k-1}|=n_{k-1}-N_{k-2}$ representations in the form $N_k=a+a'$ with $a, a'\in A$. Since $n_{k-1}-N_{k-2}>p$ for large k, it follows that $N_k\in 2(A\setminus S)$ for all sufficiently large odd integers k.

Let $T\subset B=N\setminus A$ with $|T|=p-1$. Let k be an even integer such that $S\cup T\subset[0, N_{k-2}]$. Let S' be the set of those $a\in \bigcup_{j=1}^{k-1}(A_j\cup A''_j)\cup\{0\}$ such that $N_k-a\in A''_k$. Then $N_k\notin 2(A\setminus S)$ if and only if $S'\subset S$. If $S'\subset S$ and $S'\neq S$, then $|S'|\leq p-1$. From the construction of A''_k it follows that A''_k contains all but at most $p-2$ of the integers of the form N_k-b with $b\in \bigcup_{j=1}^{k-2}(B_j\cup B''_j)$. Therefore, if $T\subset \bigcup_{j=1}^{k-2}(B_j\cup B''_j)$ and if $|T|=p-1$, then $N_k\in 2((A\setminus S)\cup T)$.

Suppose that $S' = S$. Let T' be the set of those $b \in \bigcup_{j=1}^{k-2} (B_j \cup B''_j)$ such that $N_k - b \notin A'''_k$. Then $|T'| = p - 1$ by the construction of A'''_k, and $N_k \in 2((A \setminus S) \cup T)$ if and only if $T' \neq T$. However, since the pair of sets S, T was used to construct the partition $I'''_k = A'''_k \cup B'''_k$ for infinitely many even integers k, it will happen for infinitely many even k that $S = S'$ and $T = T'$, and so $N_k \notin 2((A \setminus S) \cup T)$. Therefore, $(A \setminus S) \cup T$ is a nonbasis if $|T| < |S|$.

On the other hand, if $|T| \geq p = |S|$, then $T' \neq T$ and $N_k \in 2((A \setminus S) \cup T)$. Therefore, $(A \setminus S) \cup T$ is a basis if $|S| \leq |T|$. This proves that A is an infinitely oscillating basis.

Since the sets A and $B \cup \{0\}$ were constructed by the same method, it follows that $B \cup \{0\}$ is also an infinitely oscillating basis. But $0 \notin B$, and so B is an infinitely oscillating nonbasis. This proves the Theorem.

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REFERENCES

[1] P. Erdös and M. B. Nathanson, *Maximal asymptotic nonbases*, Proc. Amer. Math. Soc. *48* (1975), 57–60.

[2] P. Erdös and M. B. Nathanson, *Oscillations of bases for the natural numbers*, Proc. Amer. Math. Soc., *53* (1975), 253–258.

[3] E. Härtter, *Ein Beitrag zur Theorie der Minimalbasen*, J. Reine Angew. Math *196* (1956), 170–204.

[4] M. B. Nathanson, *Minimal bases and maximal nonbases in additive number theory*, J. Number Theory *6* (1974), 324–333.

[5] A. Stöhr, *Gelöste und ungelöste Fragen über Basen der natürlichen Zahlenreihe, I*, J. Reine Angew. Math. *194* (1955), 40–65.

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