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Partition Theorems for Subsets of Vector Spaces

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1. INTRODUCTION

In [1] two of us investigated the problem of determining those cardinals $\alpha, \beta, \gamma, \delta, \lambda$ for which the following statement, abbreviated $\psi(\delta, \beta, \alpha, \lambda, \gamma)$, holds: "Whenever V is an α-dimensional vector space over a field of λ elements, and the δ-dimensional subspaces of V are partitioned into γ classes, there is some β-dimensional subspace of V all of whose δ-dimensional subspaces are in the same class."

In this paper, we investigate the related question of which cardinals α, β, γ, and δ make the following statement valid: "Whenever V is an α-dimensional vector space over $GF(2)$ and $V = \bigcup_{\sigma<\gamma} A_\sigma$, there are some $U \in [V]^\beta$ (the set of β-element subsets of V) and some $\sigma < \gamma$ such that if $1 \leqslant t < \delta$ and $W \in [U]^t$, then $\sum W \in A_\sigma$." This statement will be abbreviated $\langle\alpha\rangle \to \langle\beta\rangle_\gamma^\delta$. (We could of course ask the same question with a field of λ elements replacing $GF(2)$. However, we have no interesting results when $\lambda \neq 2$.) Note that the statement $\langle\alpha\rangle \to \langle\beta\rangle_\gamma^\delta$ only makes sense if $\delta \leqslant \omega$.

The statement $\langle\alpha\rangle \to \langle\beta\rangle_\gamma^\delta$ has a simple set-theoretic formulation in terms of the symmetric difference, Δ, of two sets. We will use the notation $\Delta_{i=1}^t A_i = (\Delta_{i=1}^{t-1} A_i)\,\Delta A_t$. We also take a cardinal to be the least ordinal

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of a given equipotence class. In particular, we write ω for the first infinite cardinal. $\langle\alpha\rangle \to \langle\beta\rangle_\gamma^\delta$ is equivalent to the statement "Whenever $[\alpha]^{<\omega} = \bigcup_{\sigma<\gamma} A_\sigma$ there are some $B \in [[\alpha]^{<\omega}]^\beta$ and some $\sigma < \gamma$ such that, if $1 \leqslant t < \delta$ and $\{C_i\}_{i=1}^t \in [B]^t$, then $\Delta_{i=1}^t C_i \in A_\sigma$." (The equivalence can be seen by taking $V = \{x \in \{0, 1\}^\alpha : |\{\eta \in \alpha : x(\eta) = 1\}| < \omega\}$ and associating each element x of V with $x^{-1}(\{1\})$. In this case, $x + y$ is associated with $x^{-1}(\{1\}) \,\Delta\, y^{-1}(\{1\})$.)

Under the assumption of the generalized continuum hypothesis and the nonexistence of inaccessible cardinals greater than ω, we have been able to determine the validity of $\langle\alpha\rangle \to \langle\beta\rangle_\gamma^\delta$ for given β, δ, and γ for all except at most finitely many values of α.

Section 2 consists of the development of the necessary results and counterexamples. The main theorems and some questions are presented in Section 3.

2. Development of Results

Throughout this paper, we assume that the results about $\langle\alpha\rangle \to \langle\beta\rangle_\gamma^\delta$ are not vacuous. Thus, we assume that $2 \leqslant \delta \leqslant \omega$, $\delta \leqslant \beta + 1$, $1 \leqslant \gamma$, $\beta \leqslant \alpha$ if $\alpha \geqslant \omega$ and $\beta \leqslant 2^\alpha$ if $\alpha < \omega$. We also note the following trivial implications:

Lemma 2.1. *Let $\alpha < \alpha'$, $\beta < \beta'$, $\gamma < \gamma'$, and $\delta < \delta'$.*

(a) *If $\langle\alpha\rangle \to \langle\beta\rangle_\gamma^\delta$, then $\langle\alpha'\rangle \to \langle\beta\rangle_\gamma^\delta$.*

(b) *If $\langle\alpha\rangle \to \langle\beta'\rangle_\gamma^\delta$, then $\langle\alpha\rangle \to \langle\beta\rangle_\gamma^\delta$.*

(c) *If $\langle\alpha\rangle \to \langle\beta\rangle_{\gamma'}^\delta$, then $\langle\alpha\rangle \to \langle\beta\rangle_\gamma^\delta$.*

(d) *If $\langle\alpha\rangle \to \langle\beta\rangle_\gamma^{\delta'}$, then $\langle\alpha\rangle \to \langle\beta\rangle_\gamma^\delta$.*

The following lemma relates the statement $\psi(1, \beta, \alpha, 2, \gamma)$ and $\langle\alpha\rangle \to \langle\beta\rangle_\gamma^\delta$.

Lemma 2.2.

(a) *If $1 \leqslant \beta < \omega$ and $\psi(1, \beta, \alpha, 2, \gamma)$, then $\langle\alpha\rangle \to \langle\beta\rangle_\gamma^{\beta+1}$.*

(b) *If $\beta \geqslant \omega$ and $\psi(1, \beta, \alpha, 2, \gamma)$, then $\langle\alpha\rangle \to \langle\beta\rangle_\gamma^\omega$.*

(c) *If $2 \leqslant \beta < \omega$ and $\langle\alpha\rangle \to \langle\beta\rangle_{\gamma+1}^{\beta+1}$, then $\psi(1, \beta, \alpha, 2, \gamma)$.*

(d) *If $\beta \geqslant \omega$ and $\langle\alpha\rangle \to \langle\beta\rangle_{\gamma+1}^\omega$, then $\psi(1, \beta, \alpha, 2, \gamma)$.*

(e) *If $2 \leqslant \beta < \omega$ and $\gamma \geqslant \omega$, then $\langle\alpha\rangle \to \langle\beta\rangle_\gamma^{\beta+1}$ and $\psi(1, \beta, \alpha, 2, \gamma)$ are equivalent.*

(f) If *$\beta \geqslant \omega$ and $\gamma \geqslant \omega$, then $\langle\alpha\rangle \to \langle\beta\rangle_\gamma^\omega$ and $\psi(1, \beta, \alpha, 2, \gamma)$ are equivalent.*

Proof. If S is a set of vectors from some vector space, we let $\langle S\rangle$ denote the subspace generated by S. We denote $\langle\{v\}\rangle$ by $\langle v\rangle$.

(a) Let V be an α-dimensional vector space over $GF(2)$ and let $V = \bigcup_{\sigma<\gamma} A_\sigma$. For each $\sigma < \gamma$, let $B_\sigma = \{\langle v\rangle : v \in A_\sigma\backslash\{0\}\}$. Then there are some β-dimensional subspace U of V and some $\sigma < \gamma$ such that whenever $v \in U\backslash\{0\}$, $\langle v\rangle \in B_\sigma$. Let W be a basis for U, then $W \in [V]^\beta$ and, if $1 \leqslant t < \beta + 1$ and $H \in [W]^t$, then $\sum H \in U\backslash\{0\}$ so $\sum H \in A_\sigma$.

The proof of (b) is obtained from the proof of (a) by replacing $\beta + 1$ with ω.

(c) Let V be an α-dimensional vector space over $GF(2)$ and let $\{\langle v\rangle : v \in V\backslash\{0\}\} = \bigcup_{\sigma<\gamma} B_\sigma$. Let $A_0 = \{0\}$ and for $\sigma < \gamma$ let $A_{\sigma+1} = \{v \in V\backslash\{0\} : \langle v\rangle \in B_\sigma\}$. (If σ is a limit ordinal, let $A_\sigma = \varnothing$.) Then there are some $W \in [V]^\beta$ and some $\sigma < \gamma + 1$ such that one has $\sum H \in A_\sigma$ whenever $H \subseteq W$ and $H \neq \varnothing$. Let $U = \langle W\rangle$. Note that $\sigma \neq 0$ since $\beta \geqslant 2$. Consequently, if $H \subseteq W$ and $H \neq \varnothing$ then $\sum H \neq 0$. Thus, W is a set of linearly independent vectors, and hence the dimension of U is β. That the 1-dimensional subspaces of U are contained in $B_{\sigma-1}$ is trivial.

(d) The proof of (d) is similar to the proof of (c).

(e) follows from (a) and (c), and (f) follows from (b) and (d).

As a consequence of Lemma 2.2 we have from [1] the following results.

LEMMA 2.3.

(a) *If* $\beta < \omega$, $\delta < \omega$, *and* $\gamma < \omega$, *then there is some least integer* $N(\beta, \gamma, \delta)$ *such that* $\langle N(\beta, \gamma, \delta)\rangle \to \langle\beta\rangle_\gamma^\delta$.

(b) *If* $\gamma < \omega$, *then* $\langle\omega\rangle \to \langle\omega\rangle_\gamma^\omega$.

(c) *If* $\beta < \omega$ *and the generalized continuum hypothesis is assumed, then for each ordinal* σ, $\langle\aleph_{\sigma+2}^{\beta-1}\rangle \to \langle\beta\rangle_{\aleph_\sigma}^{\beta+1}$.

(d) *If* $2 \leqslant \beta < \omega$ *and the generalized continuum hypothesis is assumed, then for each ordinal* σ, $\langle\aleph_{\sigma+\beta-1}\rangle \nrightarrow \langle\beta\rangle_{\aleph_\sigma}^{\beta+1}$.

Proof. In addition to Lemma 2.2 we need only note:

(a) That $\psi(1, \beta, N, 2, \gamma)$ holds for some N follows from [2, Corollary 2]. Thus $\langle N\rangle \to \langle\beta\rangle_\gamma^{\beta+1}$ holds hence $\langle N\rangle \to \langle\beta\rangle_\gamma^\delta$ holds, so there is a smallest N for which it holds.

(b) That $\psi(1, \omega, \omega, 2, \gamma)$ holds follows from [4, Corollary 3.5].

(c) That $\psi(1, \beta, \aleph_{\sigma+2}^{\beta-1}, 2, \aleph_\sigma)$ holds is Lemma 2.16 of [1].

(d) That $\sim\psi(1, \beta, \aleph_{\sigma+\beta-1}, 2, \aleph_\sigma)$ holds is 2.14 of [1]. Since $\beta \geqslant 2$, Lemma 2.2(c) applies.

The following lemma establishes part of the relationship between $\langle\alpha\rangle \to \langle\beta\rangle_\gamma^\delta$ and the arrow relation of [2]. Recall that $\alpha \to (\beta)_\gamma^\delta$ if and only if whenever $[\alpha]^\delta = \bigcup_{\sigma<\gamma} A_\sigma$, there are some $B \in [\alpha]^\beta$ and some $\sigma < \gamma$ such that $[B]^\delta \subseteq A_\sigma$. In the following we use the convention that, if $\beta \geqslant \omega$, then $\beta - 1 = \beta$.

LEMMA 2.4. *Let $\lambda = 2^\alpha$ if $\alpha < \omega$ and let $\lambda = \alpha$ if $\alpha \geqslant \omega$. If $\lambda \to (\beta)^2_\gamma$, then $\langle\alpha\rangle \to \langle\beta - 1\rangle^3_\gamma$.*

Proof. Assume $\lambda \to (\beta)^2_\gamma$. Let V be an α-dimensional vector over $GF(2)$ and note that $|V| = \lambda$. Let $V = \bigcup_{\sigma<\gamma} A_\sigma$. For each $\sigma < \gamma$, let $B_\sigma = \{\{x, y\} \in [V]^2 : x + y \in A_\sigma\}$. Then $[V]^2 = \bigcup_{\sigma<\gamma} B_\sigma$. Since $\lambda \to (\beta)^2_\gamma$, there are some $W \in [V]^\beta$ and some $\sigma < \gamma$ such that $[W]^2 \subseteq B_\sigma$. Pick $a \in W$ and let $U = \{a + b : b \in W \setminus \{a\}\}$. Then $U \in [V]^{\beta-1}$ and $U \subseteq A_\sigma$. Also if $\{a + b, a + c\} \in [U]^2$, then since $b + c \in A_\sigma$, $(a + b) + (a + c) \in A_\sigma$. Thus $\langle\alpha\rangle \to \langle\beta - 1\rangle^3_\gamma$ holds.

Lemma 2.6 is preliminary to a partial converse to Lemma 2.4. We will make use of the following notation.

DEFINITION 2.5. Let $\alpha \geqslant \omega$, let $F \in [\omega]^{<\omega}$, and let $G \in [\alpha]^p$ where $\max F < p$. Then $B(G, F) = \{\nu \in G : |\{\eta \in G : \eta < \nu\}| \in F\}$.

Thus, for example, if $F = \{0, 2\}$, $G = \{\nu_0, \nu_1, \nu_2, \nu_3\}$ and $\nu_0 < \nu_1 < \nu_2 < \nu_3$, then $B(G, F) = \{\nu_0, \nu_2\}$.

The following lemma allows us to assume that we have vectors which all have the same overlapping pattern.

LEMMA 2.6. *Let β be a regular infinite cardinal and let $\beta \leqslant \alpha$. Let $p < \omega$ and let $\mathfrak{V} \in [[\alpha]^p]^\beta$ such that, for all U and V in $\mathfrak{V}$, $|U \Delta V| = p$. Then there are some $J \in [\alpha]^{p/2}$, $F \in [p]^{p/2}$ and $\mathfrak{V}^* \in [\mathfrak{V}]^\beta$ such that, whenever U and V are distinct members of $\mathfrak{V}^*$, $U \cap V = J$ and either $B(U \Delta V, F) = U \setminus J$ or $B(U \Delta V, F) = V \setminus J$.*

Proof. Since $\mathfrak{V} \in [[\alpha]^p]^\beta$ and for every U and V in $\mathfrak{V}$, $U \Delta V \in [\alpha]^p$, we have immediately the existence of J in $[\alpha]^{p/2}$ such that, for every U and V in $\mathfrak{V}$, $U \cap V = J$.

For $U \in [\alpha]^p$ and $j < p$, let $S(U, j)$ be that element of U with j predecessors. By transfinite induction iterated $p/2$ times, we may choose $\mathfrak{V}' \in [\mathfrak{V}]^\beta$ and order $\mathfrak{V}' = \{W_\sigma\}_{\sigma<\beta}$ so that whenever $\rho < \sigma < \beta$ and $j < p$, one has $S(W_\rho, j) \leqslant S(W_\sigma, j)$. We can further assume that there is some $H \in [p]^{p/2}$ such that, for each $\sigma < \beta$, $B(W_\sigma, H) = J$.

We now claim that we can choose $\mathfrak{V}^* = \{V_\sigma\}_{\sigma<\beta}$ so that, if $\sigma < \tau < \rho < \beta$ and $i < j < p$ and $\{i, j\} \cap H = \varnothing$, then $S(V_\sigma, i) < S(V_\tau, j)$ if and only if $S(V_\tau, i) < S(V_\rho, j)$. Since $[p \setminus H]^2$ is finite, it suffices to produce for any given $\{i, j\} \in [p \setminus H]^2$, a monotonic function $f : \beta \to \beta$ so that whenever

$\sigma < \tau < \rho < \beta$ one has $S(W_{f(\sigma)}, i) < S(W_{f(\tau)}, j)$ if and only if $S(W_{f(\tau)}, i) < S(W_{f(\sigma)}, j)$.

To this end, let $\{i, j\} \in [p \backslash H]^2$ with $i < j$. There are two cases to consider. In Case 1, for each $\sigma < \beta$ there is some $\tau > \sigma$ such that $S(W_\sigma, j) < S(W_\tau, i)$. In this case, let $f(0) = 0$ and assume that $f(\sigma)$ has been defined, for each $\sigma < \tau$, so that whenever $\sigma < \rho < \tau$ one has $S(W_{f(\sigma)}, j) < S(W_{f(\rho)}, i)$. Let $\eta = \sup\{f(\sigma): \sigma < \tau\}$. By the regularity of β, $\eta < \beta$. Let $\nu > \eta$ such that $S(W_\eta, j) < S(W_\nu, i)$. Let $f(\tau) = \nu$. Now if $\sigma < \tau$, then $S(W_{f(\sigma)}, j) \leqslant S(W_\eta, j) < S(W_\nu, i) = S(W_{f(\tau)}, i)$. In Case 2, there is some $\sigma < \beta$ such that, for every $\tau > \sigma$, $S(W_\tau, i) < S(W_\sigma, j)$ (since for $\tau \neq \sigma$, $W_\tau \cap W_\sigma = J = B(W_\tau, H)$ and since $i \notin H$ and $j \notin H$, we have $S(W_\tau, i) \neq S(W_\sigma, j)$). In this case, for each $\nu < \beta$, let $f(\nu) = \sigma + \nu$. Thus, if $\nu < \tau < \beta$ one has $S(W_{f(\nu)}, i) = S(W_{\sigma+\nu}, i) < S(W_\sigma, j) < S(W_{\sigma+\tau}, j) = S(W_{f(\tau)}, j)$. Consequently, the claim is established.

In particular, we have if $\sigma < \tau < \beta$ and $\{i, j\} \in [p \backslash H]^2$, then $S(V_\sigma, i) < S(V_\tau, j)$ if and only if $S(V_0, i) < S(V_1, j)$. Let $F = \{i < p : S(V_0 \Delta V_1, i) \in V_0\}$. Now, if $\sigma < \tau < \beta$, one has $B(V_\sigma \Delta V_\tau, F) = V_\sigma \backslash J$, as desired.

LEMMA 2.7. *Let β be a regular infinite cardinal and let $\gamma \geqslant \omega$, then $\langle\alpha\rangle \to \langle\beta\rangle_\gamma^2$ if and only if $\alpha \to (\beta)_\gamma^2$.*

Proof. The necessity is Lemma 2.4. Assume $\langle\alpha\rangle \to \langle\beta\rangle_\gamma^2$ and let $[\alpha]^2 = \bigcup_{\sigma<\gamma} R_\sigma$. For each even $p < \omega$, order $[\alpha]^{p/2} = \{E(p, \nu)\}_{\nu<\alpha}$ and write $[p]^{p/2} = \{F(p, t)\}_{t<j(p)}$, where, of course, $j(p) = \binom{p}{p/2}$. For each even p and for each set $\{\sigma_t\}_{t<j(p)} \subseteq \gamma$, let $A(p, \sigma_0, \sigma_1, \ldots, \sigma_{j(p)-1}) = \{U \in [\alpha]^p$: for each $t < j(p)$, $\{\nu, \eta\} \in R_{\sigma_t}$ where $B(U, F(p, t)) = E(p, \nu)$ and $U \backslash B(U, F(p, t)) = E(p, \eta)\}$.

Now we may order $\{A(p, \sigma_0, \ldots, \sigma_{j(p)-1}): p$ is even and $\{\sigma_t\}_{t<j(p)} \subseteq \gamma\} \cup \{[\alpha]^p : p \text{ is odd}\} = \{B_\rho\}_{\rho<\gamma}$. Now $[\alpha]^{<\omega} = \bigcup_{\sigma<\gamma} B_\sigma$. So, by assumption, there are some $\mathfrak{V} \in [[\alpha]^{<\omega}]^\beta$ and some $\rho < \gamma$ such that $\mathfrak{V} \subseteq B_\rho$ and, if $\{E, F\} \in [\mathfrak{V}]^2$, then $E \Delta F \in B_\rho$. $B_\rho \neq [\alpha]^p$ for any odd p since, if $|E| = |F| = |E \Delta F| = p$, then $|E \cap F| = p/2$. Thus there are some even p and some $\{\sigma_\tau\}_{\tau<j(p)}$ such that $\beta_\rho = A(p, \sigma_0, \sigma_1, \ldots, \sigma_{j(p)-1})$.

Hence, given U and V in $\mathfrak{V}$, we have $|U \Delta V| = p$. By Lemma 2.6 there are $J \in [\alpha]^{p/2}$, $F \in [p]^{p/2}$, and $\mathfrak{V}^* \in [\mathfrak{V}]^\beta$ such that, whenever $\{U, V\} \in [\mathfrak{V}^*]^2$, one has $U \cap V = J$ and either $B(U \Delta V, F) = U \backslash J$ or $B(U \Delta V, F) = V \backslash J$. Now $F = F(p, t)$ for some $t < j(p)$.

Let $X = \{\nu < \alpha$: There is some $U \in \mathfrak{V}^*$ such that $U \backslash J = E(p, \nu)\}$. Then $X \in [\alpha]^\beta$. We claim that $[X]^2 \subseteq R_{\sigma_t}$. To this end, let $\{\nu, \eta\} \in [X]^2$ and pick U and V in $\mathfrak{V}^*$ such that $U \backslash J = E(p, \nu)$ and $V \backslash J = E(p, \eta)$. Without loss of generality, $B(U \Delta V, F) = U \backslash J$. Consequently, $(U \Delta V) \backslash B(U \Delta V, F) = V \backslash J$. But $U \Delta V \in A(p, \sigma_0, \ldots, \sigma_{j(p)-1})$ and $B(U \Delta V, F_t) = E(p, \nu)$ and $(U \Delta V) \backslash B(U \Delta V, F_t) = E(p, \eta)$, so $\{\nu, \eta\} \in R_{\sigma_t}$ as desired.

As a consequence of Lemma 2.7, we have from [2] the following results.

LEMMA 2.8. *Assume the generalized continuum hypothesis. Let* $\alpha > \omega$ *and let* $\beta < \alpha$.

(a) *If* $\gamma^+ < \alpha$, *then* $\langle\alpha\rangle \to \langle\beta\rangle^3_\gamma$.

(b) *If* $\beta \geqslant \omega$ *and* $\gamma^+ \geqslant \alpha$ *then* $\langle\alpha\rangle \nrightarrow \langle\beta\rangle^3_\gamma$

Proof. (a) Assume $\gamma^+ < \alpha$. By [2, theorem I], $\alpha \to (\beta)^2_\gamma$. Thus by Lemma 2.4, we have $\langle\alpha\rangle \to \langle\beta - 1\rangle^3_\gamma$. If $\beta \geqslant \omega$ this is $\langle\alpha\rangle \to \langle\beta\rangle^3_\gamma$. If $\beta < \omega$, then in fact $\langle\alpha\rangle \to \langle\omega\rangle^3_\gamma$ so $\langle\alpha\rangle \to \langle\beta\rangle^3_\gamma$.

(b) Assume $\beta \geqslant \omega$ and $\gamma^+ \geqslant \alpha$. By [2, Theorem I] $\alpha \nrightarrow (\omega)^2_\gamma$. Thus, by Lemma 2.7 we have that $\langle\alpha\rangle \nrightarrow \langle\omega\rangle^3_\gamma$. Since $\beta \geqslant \omega$, we have $\langle\alpha\rangle \nrightarrow \langle\beta\rangle^3_\gamma$.

The following result is needed to obtain the fact that $\langle\beta\rangle \to \langle\beta\rangle^3_\gamma$ fails if β is an infinite successor and $\gamma < \omega$.

LEMMA 2.9. *Let* β *be a regular cardinal*, $\beta > \omega$. *If* $\langle\beta\rangle \to \langle\beta\rangle^3_\gamma$, *then* $\beta \to (\beta)^2_\gamma$.

Proof. Let $[\beta]^{<\omega} = \{E(\rho)\}_{\rho<\beta}$. Assume $\langle\beta\rangle \to \langle\beta\rangle^3_\gamma$ and let $[\beta]^2 = \bigcup_{\sigma<\gamma} A_\sigma$. For each $U \in [\beta]^{<\omega}$, let $B(U) = \{\nu \in U : |\{\eta \in U : \eta < \nu\}| < |\eta \in U : \eta > \nu\}|\}$. (Thus $B(U)$ is the first half of U if $|U|$ is even.) For each $\sigma < \gamma$, let $B_\sigma = \{U : \{\nu, \eta\} \in A_\sigma$ where $B(U) = E(\nu)$ and $U \backslash B(U) = E(\eta)\}$.

By assumption, there are $\mho \in [[\beta]^{<\omega}]^\beta$ and $\sigma < \gamma$ such that $\mho \subseteq B_\sigma$ and, if $\{U, V\} \in [\mho]^2$, then $U \Delta V \in B_\sigma$. Since $\beta > \omega$ and β is regular, we may assume that there is some $p < \omega$ such that $\mho \subseteq [\beta]^p$.

We may choose a subfamily $\mho^* \in [\mho]^\beta$ and a set J such that if $\{U, V\} \in [\mho^*]^2$, then $U \cap V = J$. Ordering $\mho^*$ by the first member of $U \backslash J$ we in fact obtain $\mho' \in [\mho^*]^\beta$ so that if $\{U, V\} \in [\mho']^2$, then either $\max(U \backslash J) < \min(V \backslash J)$ or $\max(V \backslash J) < \min(U \backslash J)$. Let $X = \{\nu$: There is some $U \in \mho'$ such that $E(\nu) = U \backslash J\}$. Now, if $\{\nu, \eta\} \in [X]^2$, then we have U and V in $\mho'$ such that $U \backslash J = E(\nu)$ and $V \backslash J = E(\eta)$. Without loss of generality, $\max U \backslash J < \min V \backslash J$. Since $|U \backslash J| = |V \backslash J|$, we have $B(U \Delta V) = U \backslash J = E(\nu)$ and $(U \Delta V) \backslash B(U \Delta V) = V \backslash J = E(\eta)$. Since $U \Delta V \in B_\sigma$, $\{\nu, \eta\} \in A_\sigma$. Thus $[X]^2 \subseteq A_\sigma$.

LEMMA 2.10. *Assume the generalized continuum hypothesis and let* β *be an infinite nonlimit cardinal, then* $\langle\beta\rangle \nrightarrow \langle\beta\rangle^3_2$.

Proof. By [2, Theorem I], we have that $\beta \nrightarrow (\beta)^2_2$. Since β is a nonlimit β is regular so by Lemma 2.9, $\langle\beta\rangle \nrightarrow \langle\beta\rangle^3_2$.

The following lemma is needed to prove that $\langle\beta\rangle \not\to \langle\beta\rangle^3_2$ for nonregular limit cardinals. Its proof uses methods of [2].

LEMMA 2.11. *Let λ and μ be cardinals such that $\lambda = \mu^+ = 2^\mu$. Then there exists a collection of pairwise disjoint sets $\{\mathcal{O}_\sigma\}_{\sigma<\mu}$ such that $[\lambda]^{<\omega} = \bigcup_{\sigma<\mu} \mathcal{O}_\sigma$ and, for every $A \in [[\lambda]^{<\omega}]^\mu$ and every $B \in [[\lambda]^{<\omega}]^\lambda$ and every $\sigma < \mu$, there are some $y \in A$ and $x \in B$ such that $y \cup x \in \mathcal{O}_\sigma$.*

Proof. Let $[[\lambda]^{<\omega}]^\mu = \{\mathcal{B}_\sigma\}_{\sigma<\lambda}$ (with $\mathcal{B}_0 = [\mu]^{<\omega}$). For each $\sigma < \mu$, let $\mathcal{O}_{\sigma,0} = \varnothing$. Let $\rho < \lambda$ and assume that, for each $\eta < \rho$, we have chosen $\{\mathcal{O}_{\sigma,\eta}\}_{\sigma<\mu}$ such that:

(1) For each $\sigma < \mu$, $\mathcal{O}_{\sigma,\eta} \supseteq \bigcup_{\nu<\eta} \mathcal{O}_{\sigma,\nu}$.

(2) For each $\tau < \eta$, if $\bigcup\mathcal{B}_\tau \subseteq \eta$ and if $x \in [\lambda]^{<\omega}$ such that $\max x = \eta$, then there exists $\{y_\sigma\}_{\sigma<\mu} \subseteq \mathcal{B}_\tau$ such that, for $\sigma < \mu$, $y_\sigma \cup x \in \mathcal{O}_{\sigma,\eta}$.

(3) $\bigcup_{\sigma<\mu} \cup \mathcal{O}_{\sigma,\eta} \subseteq \eta + 1$.

(4) If $\sigma < \tau < \mu$, then $\mathcal{O}_{\sigma,\eta} \cap \mathcal{O}_{\tau,\eta} = \varnothing$

(5) If $\sigma < \mu$ and $x \in \mathcal{O}_{\sigma,\eta} \setminus \bigcup_{\nu<\eta} \mathcal{O}_{\sigma,\nu}$, then $\eta \in x$.

All conditions are easily verified when $\eta = 0$. If $\rho < \mu$, then there is no $\tau < \rho$ such that $\bigcup\mathcal{B}_\tau \subseteq \rho$. Consequently, we may let $\mathcal{O}_{\sigma,\rho} = \varnothing$ for each $\sigma < \mu$ and $\rho < \mu$. In this case (1), (3), and (4) are clearly satisfied and (2) and (5) are satisfied vacuously.

We now assume $\rho \geqslant \mu$. Let $\{x \in [\lambda]^{<\omega} : \max x = \rho\} = \{x_\iota\}_{\iota<\mu}$ and let $\{\mathcal{B}_\tau : \tau < \rho \text{ and } \bigcup\mathcal{B}_\tau \subseteq \rho\} = \{C_\nu\}_{\nu<\mu}$ (with repetition as necessary to fill out the list). Since $\mathcal{B}_0 = [\mu]^{<\omega}$, the latter set is nonempty. Order $\mu \times \mu \times \mu = \{(\sigma_\tau, \xi_\tau, \iota_\tau)\}_{\tau<\mu}$. For each $\tau < \mu$ we choose inductively $y_{\sigma_\tau,\xi_\tau,\iota_\tau} \in C_{\xi_\tau}$ so that $y_{\sigma_\tau,\xi_\tau,\iota_\tau} \cup x_{\iota_\tau} \notin \{y_{\sigma_\eta,\xi_\eta,\iota_\eta} \cup x_{\iota_\eta} : \eta < \tau\}$. This can be done since $|\{y_{\sigma_\eta,\xi_\eta,\iota_\eta} \cup x_{\iota_\eta} : \eta < \tau\}| \leqslant |\tau| < \mu$ while $|\{y \cup x_{\iota_\tau} : y \in C_{\xi_\tau}\}| = \mu$. (Note that $|x_{\iota_\tau}| < \omega$ and if $y \cup x_{\iota_\tau} = y' \cup x_{\iota_\tau}$ then $y \Delta y' \subseteq x_{\iota_\tau}$.)

For each $\sigma < \mu$, let $\mathcal{O}_{\sigma,\rho} = \bigcup_{\nu<\rho} \mathcal{O}_{\sigma,\nu} \cup \{y_{\sigma,\xi,\iota} \cup x_\iota : \xi < \mu \text{ and } \iota < \mu\}$. Condition (1) is trivially satisfied. Since, for each $\xi < \mu$ and $\iota < \mu$, $\bigcup C_\xi \subseteq \rho$ and $\max x_\iota = \rho$ we have conditions (3) and (5) satisfied. To verify condition (2) note that if $\bigcup\mathcal{B}_\tau \subseteq \rho$ and $x \in [\lambda]^{<\omega}$ such that $\max x = \rho$, then $\mathcal{B}_\tau = C_\xi$ and $x = x_\iota$ for some ξ and ι. Then $\{y_{\sigma,\xi,\iota} : \sigma < \mu\}$ is as required by condition (2). Finally, condition (4) is satisfied by the construction of $\{y_{\sigma,\xi,\iota} : \sigma < \mu, \xi < \mu, \iota < \mu\}$ and the fact that condition (4) held at previous levels.

Now, for each σ such that $0 < \sigma < \mu$ let $\mathcal{O}_\sigma = \bigcup_{\rho<\lambda} \mathcal{O}_{\sigma,\rho}$. Let $\mathcal{O}_0 = [\lambda]^{<\omega} \setminus \bigcup_{0<\sigma<\mu} \mathcal{O}_\sigma$ and note that $\mathcal{O}_0 \supseteq \bigcup_{\rho<\lambda} \mathcal{O}_{0,\rho}$. Then $\{\mathcal{O}_\sigma\}_{\sigma<\mu}$ is a collection of pairwise disjoint sets and $[\lambda]^{<\omega} = \bigcup_{\sigma<\mu} \mathcal{O}_\sigma$. Now let $A \in [[\lambda]^{<\omega}]^\mu$; let

$B \in [[\lambda]^{<\omega}]^{\lambda}$, and $\sigma < \mu$. Now $A = \mathscr{B}_\tau$ for some τ. Pick $\rho' < \mu$ such that $\tau < \rho'$ and $\sup \bigcup \mathscr{B}_\tau < \rho'$. Since $|B| = \lambda$, there is some $x \in B$ such that $\max x > \rho'$. Let $\rho = \max x$. Then $\bigcup \mathscr{B}_\tau \subseteq \rho$ and $\max x = \rho$ so, by condition 2, there is some $y_\sigma \in \mathscr{B}_\tau = A$ such that $y_\sigma \cup x \in \mathscr{A}_\sigma$.

LEMMA 2.12. *Let λ and μ be cardinals such that $\lambda = \mu^+ = 2^\mu$. Then there are disjoint sets $\mathscr{B}_0$ and $\mathscr{B}_1$, such that $[\lambda]^{<\omega} = \mathscr{B}_0 \cap \mathscr{B}_1$, and whenever $B \in [[\lambda]^{<\omega}]^{\lambda}$ there is $\{x_0, x_1, y_0, y_1\} \subseteq B$ such that $x_0 \cup y_0 \in \mathscr{B}_0$ and $x_1 \cup y_1 \in \mathscr{B}_1$.*

Proof. Let $\{\mathscr{A}_\sigma\}_{\sigma<\mu}$ be as guaranteed by Lemma 2.11 and let $\mathscr{B}_0 = \mathscr{A}_0$ and $\mathscr{B}_1 = \bigcup_{0<\sigma<\mu} \mathscr{A}_\sigma$. Let $A \in [B]^\mu$ and pick $\{x_0, x_1\} \subseteq B$ and $\{y_0, y_1\} \subseteq A$ as guaranteed by Lemma 2.11.

LEMMA 2.13. *Let λ and β be cardinals such that $\omega < \lambda = \mathrm{cf}(\beta) < \beta$ and let $\{\psi_\nu\}_{\nu<\lambda}$ be cofinal in β. Let $A \in [[\beta]^{<\omega}]^\beta$. Then there are some $B \subseteq \beta$ and some $\{V_\sigma\}_{\sigma<\lambda} \subseteq A$ such that $|\{\nu < \lambda$: There is some $\sigma < \lambda$ such that $(V_\sigma \setminus B) \cap [\psi_\nu, \psi_{\nu+1}) \neq \varnothing\}| = \lambda$ and whenever $\sigma < \tau < \lambda$ one has $V_\sigma \cap V_\tau = B$.*

Proof. Since $\mathrm{cf}(\beta) > \omega$, we may assume that there is some $t < \omega$ such that $A \subseteq [\beta]^t$. We may choose inductively $\{W_\sigma\}_{\sigma<\lambda}$ such that $|\{\nu < \lambda$: There is some $\sigma < \lambda$ such that $W_\sigma \cap [\psi_\nu, \psi_{\nu+1}) \neq \varnothing\}| = \lambda$. (We use the fact that, for $\nu < \lambda$, $|[\psi_\nu]^t| = |\psi_\nu|$.) Then, since λ is regular, we may find B and $\{V_\sigma\}_{\sigma<\lambda} \subseteq \{W_\sigma\}_{\sigma<\lambda}$ such that $V_\sigma \cap V_\tau = B$ when $\sigma < \tau < B$.

LEMMA 2.14. *Let β be a cardinal such that $\mathrm{cf}(\beta) < \beta$ and $\mathrm{cf}(\beta) = \mu^+ = 2^\mu$ for some cardinal μ, then $\langle\beta\rangle \nrightarrow \langle\beta\rangle^3_2$.*

Proof. Let $\lambda = \mathrm{cf}(\beta)$ and let $\{\psi_\nu\}_{\nu<\lambda}$ be cofinal in β. Let $\mathscr{B}_0$ and $\mathscr{B}_1$ be as guaranteed by Lemma 2.12. For $i < 2$, let $A_i = \{V \in [\beta]^{<\omega} : \{\nu < \lambda: V \cap [\psi_\nu, \psi_{\nu+1}) \neq \varnothing\} \in \mathscr{B}_i\}$, then $[\beta]^{<\omega} = A_0 \cup A_1$. Suppose there are some $i < 2$ and $\mathfrak{V} \in [[\beta]^{<\omega}]^\beta$ such that $\mathfrak{V} \subseteq A_i$ and, whenever $\{V, W\} \in [\mathfrak{V}]^2$, $V \Delta W \in A_i$.

Choose $\{V_\sigma\}_{\sigma<\lambda} \subseteq \mathfrak{V}$ and B as guaranteed by Lemma 2.13. Let, for each $\sigma < \lambda$, $K_\sigma = \{\nu < \lambda: [\psi_\nu, \psi_{\nu+1}) \cap (V_\sigma \setminus B) \neq \varnothing\}$. By Lemma 2.13. $|\{\nu < \lambda : \nu \in K_\sigma$ for some $\sigma\}| = \lambda$. Since each K_σ is finite, we have $|\{K_\sigma : \sigma < \lambda\}| = \lambda$. Thus, by Lemma 2.12, there are σ and τ less than λ such that $K_\sigma \cup K_\tau \notin \mathscr{B}_i$. But $K_\sigma \cup K_\tau = \{\nu < \lambda : [\psi_\nu, \psi_{\nu+1}) \cap (V_0 \Delta V_\tau) \neq \varnothing\}$, thus $V_\sigma \Delta V_\tau \notin A_i$, which is a contradiction.

In the presence of the generalized continuum hypothesis and the absence of inaccessible cardinals bigger than ω, Lemmas 2.10 and 2.14 show that $\langle\beta\rangle \rightarrow \langle\beta\rangle^3_2$ fails except possibly when $\mathrm{cf}(\beta) = \omega$. If $\beta = \omega$, then by

Lemma 2.3 we have $\langle\beta\rangle \to \langle\beta\rangle_\gamma^\omega$ for $\gamma < \omega$. The following lemma establishes that $\langle\beta\rangle \to \langle\beta\rangle_\gamma^3$ holds if $\gamma < \omega = \mathrm{cf}(\beta) < \beta$.

LEMMA 2.15. *Let $\gamma < \omega$ and let $\beta > \mathrm{cf}(\beta) = \omega$, then $\langle\beta\rangle \to \langle\beta\rangle_\gamma^3$.*

Proof. Let $\{\mu_t\}_{t<\omega}$ be a set of cardinals cofinal in β, and assume that $\mu_t < \mu_s$ when $t < s$. Let $[\beta]^{<\omega} = \bigcup_{\sigma<\gamma} A_\sigma$. Using [2, Lemma 3], in a fashion similar to the proof of [2, Lemma 3B], we obtain disjoint sets $\{S_t\}_{t<\omega}$ with the following properties:

(1) For each $t < \omega$, $|S_t| = \mu_t$.

(2) $\bigcup_{t<\omega} S_t \subseteq \beta$.

(3) $\{A_\sigma\}_{\sigma<\gamma}$ is supercanonical in $\{S_t\}_{t<\omega}$, for sets of size $2^{\gamma+1}$ or less. That is if $r \leqslant 2^{\gamma+1}$ and $A \in [\bigcup_{t<\omega} S_t]^r$ and $B \in [\bigcup_{t<\omega} S_t]^r$ and $\{t : A \cap S_t \neq \varnothing\} = \{t_0, t_1, \ldots, t_p\}$, with $t_0 < t_1 < \cdots < t_p$, and $\{t : B \cap S_t \neq \varnothing\} = \{s_0, s_1, \ldots, s_p\}$, with $s_0 < s_1 < \cdots < s_p$, and for each i, with $0 \leqslant i \leqslant p$, $|A \cap S_{t_i}| = |B \cap S_{s_i}|$, then A and B are in the same cell of $\{A_\sigma\}_{\sigma<\gamma}$.

In particular, we have that if $r \leqslant 2^{\gamma+1}$, $A \in [\bigcup_{t<\omega} S_t]^r$, $B \in [\bigcup_{t<\omega} S_t]^r$, for each $t < \omega$, $|A \cap S_t| \in \{0, 2\}$ and $|B \cap S_t| \in \{0, 2\}$, and $|\{t < \omega : A \cap S_t \neq \varnothing\}| = |\{t < \omega : B \cap S_t \neq \varnothing\}|$, then A and B are in the same cell of $\{A_\sigma\}_{\sigma<\gamma}$. By the pigeon hole principle, there exist $j < r \leqslant \gamma$ and $\sigma < \gamma$ such that, whenever $A \in [\bigcup_{t<\omega} S_t]^{<2^{\gamma+1}+1}$ and, for each $t < \omega$, $|A \cap S_t| \in \{0, 2\}$ and $|\{t < \omega : A \cap S_t \neq \varnothing\}| \in \{2^j, 2^r\}$, then $A \in A_\sigma$.

Write, for each $t < \omega$, $S_t = \{d(t, \rho)\}_{\rho<\mu_t}$. For each $t < \omega$, and each ρ such that $1 < \rho < \mu_t$, let

$$\begin{aligned} B(t, \rho) = {} & \{d(p, 0) : p < 2^{r-1} \text{ or } t + 2^{r-1} \leqslant p < t + 2^r\} \\ & \cup \{d(p, 1) : p < 2^{r-1} \text{ or } t + 2^{r-1} \leqslant p < t + 2^r - 2^j\} \\ & \cup \{d(p, \rho) : t + 2^r - 2^j \leqslant p < t + 2^r\}. \end{aligned}$$

Let $\mathfrak{W} = \{B(h \cdot 2^r, \rho) : h < \omega \text{ and } 1 < \rho < \mu_{h \cdot 2^r}\}$. Then $|\mathfrak{W}| = \beta$, and, for $h < \omega$ and ρ such that $1 < \rho < \mu_{h\cdot 2^r}$, we have $|\{t < \omega : B(h \cdot 2^r, \rho) \cap S_t \neq \varnothing\}| = 2^r$. Thus, since for each $h < \omega$ and $1 < \rho < \mu_{h\cdot 2^r}$ and each $t < \omega$, $|B(h \cdot 2^r, \rho) \cap S_t| \in \{0, 2\}$, we have $\mathfrak{W} \subseteq A_\sigma$. Now let $\{B(h \cdot 2^r, \rho), B(q \cdot 2^r, \nu)\} \in [\mathfrak{W}]^2$. If $h = q$ then $\nu \neq \rho$ and $|\{t < \omega : (B(h \cdot 2^r, \rho) \,\Delta\, B(q \cdot 2^r, \nu)) \cap S_t \neq \varnothing\}| = 2^j$. If $h \neq q$ then $|\{t < \omega : (B(h \cdot 2^r, \rho) \,\Delta\, B(q \cdot 2^r, \nu)) \cap S_t \neq \varnothing\}| = 2^r$. In either case, for every $t < \omega$, $|(B(h \cdot 2^r, \rho) \,\Delta\, B(q \cdot 2^r, \nu)) \cap S_t| \in \{0, 2\}$. Thus $B(h \cdot 2^r, \rho) \,\Delta\, B(q \cdot 2^r, \nu) \in A_\sigma$, as desired.

This result completes our available information about $\langle\alpha\rangle \to \langle\beta\rangle_\gamma^3$. The following result of Schur [6] (see also [7]) will be useful for Lemma 2.17. Note that $[r!e] = \sum_{i=0}^r r!/i!$.

LEMMA 2.16. *(Schur) Let* $0 < r < \omega$. *If* $\{1, 2, ..., [r!e]\} = \bigcup_{\sigma<r} A_\sigma$, *then there are some* x *and* y *such that* $\{x, y, x+y\} \subseteq A_\sigma$.

LEMMA 2.17. *Let* $2 \leqslant \gamma < \omega$, *let* $\beta = \aleph_\xi$, *let* $t(\gamma) = 2[\gamma!e] - 1$. *Then* $\langle \aleph_{\xi+t(\gamma)} \rangle \to \langle \beta \rangle_\gamma^4$.

Proof. Let $\alpha = \aleph_{\xi+t(\gamma)}$. Let V be an α-dimensional vector space over $GF(2)$, and let $V = \bigcup_{\sigma<\gamma} A_\sigma$. For each $\sigma < \gamma$, let $B_\sigma = \{D \in [V]^{<\omega} : \sum D \in A_\sigma\}$. Write $V = \{v_\eta\}_{\eta<\alpha}$.

Let $P = \Pi_{i=1}^{t(\gamma)+1} \gamma$. For each $s \in P$, let $C_s = \{\{v_{\eta_1}, v_{\eta_2}, ..., v_{\eta_{t(\gamma)+1}}\} \in [V]^{t(\gamma)+1} : \eta_1 < \eta_2 < \cdots < \eta_{t(\gamma)+1}$ and, for every $q \leqslant t(\gamma)+1$, $\{v_{\eta_1}, v_{\eta_2}, ..., v_{\eta_q}\} \in B_{s_q}\}$. Then $[V]^{t(\gamma)+1} = \bigcup_{s\in P} C_s$. By [2, Theorem I], we have $\alpha \to (\beta)_n^{t(\gamma)+1}$, where $n = \gamma^{t(\gamma)+1}$. (This portion of [2, Theorem I] is due independently to Kurepa [5].) Consequently, noting that $|P| = n$, there are some $U \in [V]^\beta$ and some $s \in P$ such that $[U]^{t(\gamma)+1} \subseteq C_s$. Note that, if $q \leqslant t(\gamma)+1$, then $[U]^q \subseteq B_{s_q}$. (To see this, let $\{v_{\eta_1}, v_{\eta_2}, ..., v_{\eta_q}\} \in [U]^q$ with $\eta_1 < \eta_2 \cdots < \eta_q$. Pick $\eta_{q+1} < \cdots < \eta_{t(\gamma)+1}$, with $\eta_q < \eta_{q+1}$. Then $\{v_{\eta_1}, v_{\eta_2}, ..., v_{\eta_{t(\gamma)+1}}\} \in C_s$ so $\{v_{\eta_1}, ..., v_{\eta_q}\} \in B_{s_q}$.) Let for $\sigma < \gamma$, $D_\sigma = \{q : 1 \leqslant q \leqslant [\gamma!e]$ and $s_{2q} = \sigma\}$. Then $\{1, 2, ..., [\gamma!e]\} = \bigcup_{\sigma<\gamma} D_\sigma$, so by Schur's theorem (Lemma 2.16), we may find x, y, and $\sigma < \gamma$, such that $\{x, y, x+y\} \subseteq D_\sigma$ and $x \leqslant 2y$. Let $z = 2y - x$.

Let $F \in [U]^z$ and let $U \backslash F = \bigcup_{\rho<\beta} S_\rho$ where, for each $\rho < \beta$, $|S_\rho| = x$ and $\{S_\rho\}_{\rho<\beta}$ is a pairwise disjoint collection. Let $T = \{\sum F + \sum S_\rho : \rho < \beta\}$. Now $|T| = \beta$. If $u \in T$, then for some ρ, $u = \sum (F \cup S_\rho)$ and $|F \cup S_\rho| = x + z = 2y$ while $s_{2y} = \sigma$. That is, $F \cup S_\rho \in B_\sigma$ so $u \in A_\sigma$.

Next let $\{u, v\} \in [T]^2$. Then $u + v = \sum S_\rho + \sum S_\mu = \sum (S_\rho \cup S_\mu)$ for some $\rho < \mu < \beta$. Now $|S_\rho \cup S_\mu| = 2x$, while $s_{2x} = \sigma$. Thus, $S_\rho \cup S_\mu \in B_\sigma$ so $u + v \in A_\sigma$.

Finally, let $\{u, v, w\} \in [T]^3$. Then $u + v + w = \sum (F \cup S_\rho \cup S_\mu \cup S_\nu)$ for some $\rho < \mu < \nu < \beta$. Now $|F \cup S_\rho \cup S_\mu \cup S_\nu| = 2y + 2x$ while $S_{2y+2x} = \sigma$. Thus, $F \cup S_\rho \cup S_\mu \cup S_\nu \in B_\sigma$ so $u + v + w \in A_\sigma$.

LEMMA 2.18. *For any cardinal* α *(with* $\alpha \geqslant \omega$*)*, $\langle \alpha \rangle \nrightarrow \langle \omega \rangle_\omega^4$.

Proof. $[\alpha]^{<\omega} = \bigcup_{\sigma<\omega} [\alpha]^\sigma$. Suppose we have some $\mathfrak{w} \in [[\alpha]^{<\omega}]^\omega$ and some $\sigma < \omega$ such that, whenever $\{U, V, W\} \in [\mathfrak{w}]^3$ we have $\{U, U \Delta V, U \Delta V \Delta W\} \subseteq [\alpha]^\sigma$. Since for any $\{U, V\} \in [\mathfrak{w}]^2$, we have $|U \Delta V| = \sigma$ and $|U| = \sigma$, we have some J such that $|J| = \sigma/2$ and, for any $\{U, V\} \in [\mathfrak{w}]^2$, $U \cap V = J$. But then letting U, V and W be any three members of $\mathfrak{w}$ we have $|U \Delta V \Delta W| = 2\sigma$, a contradiction.

Our final lemma stands in contrast to Lemma 2.17, establishing that if $\langle \alpha \rangle \to \langle \beta \rangle_2^5$ holds, then $\beta \leqslant \omega$.

LEMMA 2.19. *For any cardinal* α (*with* $\alpha \geqslant \aleph_1$), $\langle\alpha\rangle \not\rightarrow \langle\aleph_1\rangle_2^4$

Proof. Let $A_0 = \bigcup_{t<\omega} \bigcup_{p<\omega} [\alpha]^{2^{2t}(2p+1)}$ and let $A_1 = \bigcup_{t<\omega} \bigcup_{p<\omega} [\alpha]^{2^{2t+1}(2p+1)}$. (Thus, for $B \in [\alpha]^{<\omega}$, $B \in A_0$ if and only if $|B|$ has an even number of factors of 2.) Suppose we have $\sigma < 2$ and $\mathfrak{w} \in [[\alpha]^{<\omega}]^{\aleph_1}$, such that, whenever $\{T, U, V, W\} \in [\mathfrak{w}]^4$ we have $\{T, T\Delta U, T\Delta U\Delta V, T\Delta U\Delta V\Delta W\} \subseteq A_\sigma$. Without loss of generality there is some $t < \omega$ such that $\mathfrak{w} \subseteq [\alpha]^t$. There are $\mathfrak{w}' \in [\mathfrak{w}]^{\aleph_1}$, and $J \in [\alpha]^{<\omega}$ such that, whenever $\{U, V\} \in [\mathfrak{w}']^2$, we have $U \cap V = J$. Now let $\{T, U, V, W\} \in [\mathfrak{w}']^4$ and let $j = |J|$. Then $|T\Delta U| = 2(t-j)$ and $|T\Delta U\Delta V\Delta W| = 4(t-j)$. Thus, $T\Delta U \in A_0$ if and only if $T\Delta U\Delta V\Delta W \in A_1$, a contradiction.

3. THE MAIN THEOREM

As previously remarked, we have, under the assumption of the generalized continuum hypothesis and the nonexistence of inaccessible cardinals (that is regular limit cardinals) bigger than ω, been able to determine the validity of $\langle\alpha\rangle \rightarrow \langle\beta\rangle_\gamma^\delta$ for all but possibly finitely many values of α, given any β, γ, and δ. The exclusions of the hypothesis of Theorem 3.1 describe these unknown values, except for the case β, γ, and δ are all finite. In this case, the value of $N(\beta, \gamma, \delta)$ is not known, although crude bounds can be determined from [3]. Recall that we have assumed $\gamma \geqslant 1$, $\delta \geqslant 2$, $\delta \leqslant \beta + 1$ if $\beta < \omega$, $\delta \leqslant \omega$ if $\beta \geqslant \omega$, $\beta \leqslant 2^\alpha$ if $\alpha < \omega$, and $\beta \leqslant \alpha$ if $\alpha \geqslant \omega$.

THEOREM 3.1. *Assume the generalized continuum hypothesis and assume that there do not exist inaccessible cardinals greater than* ω. *Exclude the possibility that any of the conditions* (a), (b) *or* (c) *holds*:

(a) $\delta < \omega$, $\beta < \omega$, $\gamma = \aleph_\rho$, *and* $\aleph_{\rho+\delta-2} \leqslant \alpha < \aleph_{\rho+2}^{\beta-1}$;

(b) $\delta = 4$, $\gamma < \omega$, $\beta = \aleph_\rho > \mathrm{cf}(\beta) = \omega$, *and* $\alpha < \aleph_{\rho+t(\gamma)}$;

(c) $\delta = 4$, $\gamma < \omega$, $\mathrm{cf}(\beta) > \omega$, $\beta = \aleph_\rho$, *and* $\aleph_\rho < \alpha < \aleph_{\rho+t(\gamma)}$.

($N(\beta, \gamma, \delta)$ *and* $t(\gamma)$ *are as in Lemmas* 2.3(a) *and* 2.17 *respectively*).

Then $\langle\alpha\rangle \rightarrow \langle\beta\rangle_\gamma^\delta$ *holds if and only if one of the following* 10 *statements holds*.

(1) $\gamma = 1$;

(2) $\delta = 2$ *and* $\beta = 1$;

(3) $\delta = 2$, $\alpha \geqslant \omega$, $\gamma < \alpha$, *and* $\beta < \alpha$;

(4) $\delta = 2$, $\alpha \geqslant \omega$, $\gamma < \mathrm{cf}(\alpha)$, *and* $\beta = \alpha$;

(5) $\delta = 3$, $\beta < \alpha$, $\alpha > \omega$, *and* $\gamma^+ < \alpha$;

(6) $\delta = 3$, $\beta = \alpha > \omega$, $\mathrm{cf}(\alpha) = \omega$, *and* $\gamma < \omega$;

(7) $\delta = 4$, $\gamma < \omega$, $\beta = \aleph_\rho$, *and* $\alpha \geqslant \aleph_{\rho+t(\gamma)}$;

(8) $\beta < \omega$, $\gamma < \omega$, $\alpha \geqslant N(\beta, \gamma, \delta)$;

(9) $\beta < \omega$, $\gamma = \aleph_\rho$, *and* $\alpha \geqslant \aleph_{\rho+2^{\beta-1}}$;

(10) $\beta = \omega$ *and* $\gamma < \omega$.

Proof. That each of statements (1) through (4) is sufficient for $\langle\alpha\rangle \to \langle\beta\rangle^\delta_\gamma$ is trivial. That each of the statements (5), (6), (7), (8), (9), and (10) imply $\langle\alpha\rangle \to \langle\beta\rangle^\delta_\gamma$ follows from Lemmas 2.8(a), 2.15, 2.17, 2.3(a), 2.3(c), and 2.3(b), respectively.

Now assume that each of the statements (1) through (10) fails and $\langle\alpha\rangle \to \langle\beta\rangle^\delta_\gamma$ holds. Since statement (1) fails we have $\gamma \geqslant 2$.

We claim next that $\delta \geqslant 3$. Suppose instead that $\delta = 2$, and note that, since (2) fails, $\beta \geqslant 2$. If $\alpha < \omega$ we must have, by the pigeon hole principle, $\gamma \leqslant 2^\alpha$ and hence, since (8) fails, that $\alpha < N(\beta, \gamma, \delta)$. But this contradicts the choice of $N(\beta, \gamma, \delta)$ as the least value for which $\langle N(\beta, \gamma, \delta)\rangle \to \langle\beta\rangle^\delta_\gamma$ held. Thus $\alpha \geqslant \omega$. But then, since $\beta \geqslant 2$ and $\langle\alpha\rangle \to \langle\beta\rangle^\delta_\gamma$ holds, we must have trivially either (3) or (4) holding. This contradiction establishes that $\delta \geqslant 3$.

We claim next that $\beta \geqslant \omega$. Indeed, suppose $\beta < \omega$. Then we have, since (9) fails, either $\gamma < \omega$ or both $\gamma = \aleph_\rho$ and $\alpha < \aleph_{\rho+2^{\beta-1}}$. Suppose $\gamma = \aleph_\rho$ and $\alpha < \aleph^{\beta-1}_{\rho+2}$. Then by exclusion (a), we have $\alpha \leqslant \aleph_{\rho+\delta-2}$. Since $\delta \geqslant 3$, we have, by Lemma 2.3(d), that $\langle\aleph_{\rho+\delta-2}\rangle \not\to \langle\delta - 1\rangle^\delta_{\aleph_\rho}$. And, since $\beta \geqslant \delta - 1$, we have $\langle\alpha\rangle \not\to \langle\beta\rangle^\delta_\gamma$, a contradiction. Thus we have $\gamma < \omega$. Since (8) fails, and since $\beta < \omega$, we have $\alpha < N(\beta, \gamma, \delta)$ again contradicting the choice of $N(\beta, \gamma, \delta)$. Thus, $\beta \geqslant \omega$ as claimed.

Next we claim that $\delta \leqslant 4$. Suppose instead $\delta \geqslant 5$. Then by Lemma 2.19, $\beta \leqslant \omega$ and hence $\beta = \omega$. Since (10) fails, $\gamma \geqslant \omega$. But then, by Lemma 2.18, $\langle\alpha\rangle \not\to \langle\beta\rangle^\delta_\gamma$, a contradiction.

We thus have that $\gamma \geqslant 2$, $\beta \geqslant \omega$, and $3 \leqslant \delta \leqslant 4$. Suppose now that $\delta = 4$. By Lemma 2.18, we have $\gamma < \omega$. Since (10) fails, $\beta > \omega$. Since (7) fails, we must have $\alpha < \aleph_{\rho+t(\gamma)}$, where $\beta = \aleph_\rho$. Since we assume that no inaccessible cardinals bigger than ω exist, there are three cases to consider:

(i) β is a limit cardinal and $\beta > \mathrm{cf}(\beta) = \omega$;

(ii) β is a limit cardinal and $\beta > \mathrm{cf}(\beta) > \omega$;

(iii) β is a successor.

Case (i) is impossible by exclusion (b). In either of the cases (ii) or (iii) we have, by exclusion (c), that $\alpha = \beta$. But then, case (ii) is impossible by Lemma 2.14 (note that $\mathrm{cf}(\beta)$ is regular and hence, under our assumptions, a successor). Case (iii) is impossible by Lemma 2.10.

Thus $\delta = 3$. Since (5) fails, we have either $\alpha = \beta$ or both $\beta < \alpha$ and $\gamma^+ \geqslant \alpha$. But by Lemma 2.8(b), this latter alternative is impossible. Thus we have $\alpha = \beta$.

We claim $\beta > \omega$. Indeed, if $\beta = \omega$, then $\gamma \geqslant \omega$ since (10) fails. But, we have $\omega \nrightarrow (\omega)^{\omega}_{\omega}$ so, by Lemma 2.7 we have $\langle\beta\rangle \nrightarrow \langle\beta\rangle^3_\gamma$ a contradiction. Since (6) fails, we have either $\mathrm{cf}(\beta) > \omega$ or both $\mathrm{cf}(\beta) = \omega$ and $\gamma \geqslant \omega$. In the former case we have, by Lemma 2.10 or 2.14 depending on whether β is a successor or a limit, $\langle\alpha\rangle \nrightarrow \langle\beta\rangle^3_\gamma$ which is a contradiction. Thus, $\mathrm{cf}(\beta) = \omega$ and $\gamma \geqslant \omega$. But then $\langle\beta\rangle \nrightarrow \langle\beta\rangle^2_\gamma$, so $\langle\beta\rangle \nrightarrow \langle\beta\rangle^3_\gamma$. This is a contradiction, and the proof is complete.

There are several obvious questions arising from the exclusions of Theorem 3.1 as well as its assumption of the generalized continuum hypothesis and of the nonexistence of inaccessible cardinals greater than ω. Of particular interest, in view of the fact that, under the above assumptions, $\langle\beta\rangle \nrightarrow \langle\beta\rangle^3_2$ when $\mathrm{cf}(\beta) > \omega$, is the following question.

3.2. QUESTION. Does $\langle\beta\rangle \rightarrow \langle\beta\rangle^4_\gamma$ when $\gamma < \omega$ and $\beta > \mathrm{cf}(\beta) = \omega$?

The statement $\langle\alpha\rangle \rightarrow \langle\beta\rangle^\delta_\gamma$ can, in its set theoretic version, be restated with "Δ" replaced by "$\bigcup$." We have not attempted to deal with this question, since our interest arose from the algebraic statement.

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