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Denominators of Egyptian Fractions

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I. INTRODUCTION AND NOTATION

A fraction a/b is said to be written in Egyptian form if we write

$$\frac{a}{b} = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k}, \qquad n_1 < n_2 < \cdots < n_k,$$

where the n_i are integers. The problem of existence of such an expansion was settled in 1202 by Fibonacci who gave an algorithm which was rediscovered and more deeply investigated by Sylvester [7] in 1880. Since then several algorithms have been given in an attempt to find a more computable one and the one for which k is minimal. The algorithms to date may be summarized as follows:

1. The Fibonacci–Sylvester algorithm for which $k \leqslant a$ and n_i grow exponentially.
2. The algorithm given by Erdös in 1950 [3] for which $k \leqslant 8 \ln b/\ln \ln b$ and $n_k \leqslant 4b^2 \ln b/\ln \ln b$ for b large.
3. The algorithm of Golomb [4] in 1962 for which $k \leqslant a$ and $n_k \leqslant b(b-1)$.
4. The algorithm based on Farey series given by Bleicher in 1968 [1] for which $k \leqslant a$ and $n_k \leqslant b(b-1)$.
5. The algorithm based on continued fractions given by Bleicher [2] in 1972 for which $k \leqslant \min\{a, 2(\ln b)^2/\ln \ln b\}$ and $n_k \leqslant b(b-1)$.

In this paper we concentrate on giving an algorithm which minimizes n_k and relaxes the attempt to minimize k.

Let $D(a, b)$ be the minimal value of n_k in all expansions of a/b. Let $D(b)$ be given by $D(b) = \max\{D(a, b): 0 < a < b\}$. In this work we show, Theorem 2, that $D(b) \leqslant Kb(\ln b)^3$ for some constant K. On the other hand in Theorem 1 we show that for P a prime $D(P) \geqslant P\{\{\log_2 P\}\}$ where $\{\{x\}\} = -[-x]$ is the least integer not less than x. There is both theoretical and computational evidence to indicate that $D(N)/N$ is maximum when N is a prime.

For more historical details and bibliography see [1] and [2].

II. The Main Theorems

We begin by obtaining the lower bound for $D(N)$.

Theorem 1. *If P is a prime then $D(P) \geqslant P\{\{\log_2 P\}\}$, where $\{\{x\}\} = -[-x]$ is the least integer not less than x.*

Proof. If $a/P = \sum_{i=1}^{k} 1/n_i$, $n_1 < n_2 < \cdots < n_k$, then some of the n_i are divisible by P, while perhaps others are not. Let $x_1 < x_2 < \cdots < x_t$ be all those integers divisible by P which occur in an expansion with minimum n_k of a/P for $a = 1, 2,..., P-1$. Thus for each choice of a

$$\frac{a}{P} = \frac{1}{x_{i_1}} + \frac{1}{x_{i_2}} + \cdots + \frac{1}{x_{i_j}} + \frac{1}{y_1} + \cdots + \frac{1}{y_l},$$

where $P \mid x_{i_n}$ and $P \nmid y_n$. Let x_i' be defined by $x_i'P = x_i$, then $(x_i', P) = 1$ or the theorem is obviously true. It follows that

$$ax'_{i_1},..., x'_{i_j} - \sum{}^{*} x'_{i_1},..., x'_{i_{j-1}} \equiv 0 \bmod P,$$

where $\sum^{*} x'_{i_1},..., x'_{i_{j-1}}$ denotes the symmetric sum of all products of $j-1$ distinct terms from $\{x'_{i_1},..., x'_{i_j}\}$. For each of the $P-1$ choices of a we must get a different subset $\{x'_{i_1},..., x'_{i_j}\}$ of $\{x_1', x_2',..., x_t'\}$. Since there are at most $2^t - 1$ such, subsets we see that $2^t - 1 \geqslant P - 1$, whence $t \geqslant \log_2 P$. Since $x_1 < x_2 < \cdots < x_t$ and are all multiples of P, it follows that $x_t \geqslant P\{\{\log_2 P\}\}$. Since x_t occurs in some minimal expansion of a/P, the theorem follows.

We next prove some lemmas needed in our proof of an upper bound for $D(N)$.

We use P_k to denote the kth prime. In our notation $P_1 = 2$.

Definition. Let $\Pi_k = P_1 \cdot P_2 \cdots P_k$ be the product of the first k primes, with the convention that $\Pi_k = 1$ for $k \leqslant 0$.

As usual $\sigma(n)$ denotes the sum of the divisors of n.

LEMMA 1. *If* $1 \leqslant r \leqslant \sigma(\Pi_k)$ *then* r *can be written as a sum of distinct divisors of* Π_k.

Proof. The lemma is clearly true for $k = 0, 1, 2$. We proceed by induction on k. Suppose the lemma is true for $k < N$. Let $r \leqslant \sigma(\Pi_N)$. If $r \leqslant \sigma(\Pi_{N-1})$ we are done by induction. Therefore we suppose $\sigma(\Pi_{N-1}) < r \leqslant \sigma(\Pi_N)$. Since $\sigma(\Pi_N) = \sigma(\Pi_{N-1}) \cdot P_N(1 + 1/P_N) = \sigma(\Pi_{N-1})(P_N + 1)$, we see that $\sigma(\Pi_N) - \sigma(\Pi_{N-1}) = P_N\sigma(\Pi_{N-1})$. It follows that $r - \sigma(\Pi_{N-1}) \leqslant P_N\sigma(\Pi_{N-1})$. Also for $N \geqslant 3$, $r > \sigma(\Pi_{N-1}) \geqslant 2P_{N-1} > P_N$. Thus we can find a number s such that

1. $0 < r - sP_N \leqslant \sigma(\Pi_{N-1})$.
2. $0 < s \leqslant \sigma(\Pi_{N-1})$.

Thus $s = \sum d_i'$ where $d_i' \mid \Pi_{N-1}$ and the d_i' are distinct and $r - sP_N = \sum d_i$ where $d_i \mid \Pi_{N-1}$ and the d_i are distinct. But $d_i'P_N \mid \Pi_N$ while $d_i'P_N \nmid \Pi_{N-1}$. Thus

$$r = \sum (d_i'P_N) + \sum d_i$$

is a representation of r in the desired form. The lemma is proved.

LEMMA 2. *Let* P *be a prime and* k *an integer with* $0 \leqslant k < P$. *Given any* k *integers* $\{x_1, \ldots, x_k\}$ *none of which is divisible by* P *then the* 2^k *sums of subsets of* $\{x_1, \ldots, x_k\}$ *lie in at least* $k + 1$ *distinct congruence classes* mod P.

Proof. Although this lemma is known we give a proof since neither of the authors knows where to find this lemma in the literature.

The proof is by induction on k. For $k = 0$ the result is obvious. Suppose $P > k > 0$ and the result is true for fewer than k integers. From $x_1, \ldots, x_{k-1}$ form all possible sums. If there are more than k distinct sums mod P we are done if not by induction there are exactly k such sum. Add x_k to each of these sums if at least one new congruence class is obtained then there are enough distinct congruence classes. If no new congruence classes are obtained then let $x_k = x_{k+1} = x_{k+2} = \cdots = x_{k+P}$, and note that by adding each of these x_i, one at a time, we still remain at k distinct values, but this is absurd since from P values in the same class we can obtain all values mod P. The lemma follows.

We note that if we don't allow the empty sum the lemma remains true except that the number of distinct sums is reduced by one.

LEMMA 3. *If* r *is any integer satisfying* $\Pi_k(1 - 1/k) \leqslant r \leqslant \Pi_k(2 - 1/k)$ *then there are distinct divisors* d_i *of* Π_k *such that*

1. $r = \sum d_i$,

and

$$2. \quad d_i \geqslant c\Pi_{k-3},$$

for some constant c.

Proof. We choose N_0 sufficiently large that all of the inequalities in the remainder of the proof which are claimed to be true for sufficiently large N are valid for $N \geqslant N_0$. We pick c sufficiently small ($c = \Pi_{N_0-3}^{-1}$ will certainly work) that the lemma is true for $k \leqslant N_0$. This can be done by Lemma 1, since $\sigma(\Pi_k) \geqslant \Pi_k(2 - 1/k)$ for $k \geqslant 1$; while $k \leqslant 0$ can be handled trivially.

We proceed by induction suppose $N > N_0$ and the lemma is true for $k < N$. Let $\Pi_N(1 - 1/N) \leqslant r \leqslant \Pi_N(2 - 1/N)$.

Step I. Let $\mathscr{D}$ be the set of divisors of Π_N defined as follows $\mathscr{D} = \{d: d = \Pi_{N-1}/P_iP_jP_k,\ [N/2] \leqslant i < j < k < N\}$, when $[x]$ is the greatest integer in x. Since $|\mathscr{D}| \geqslant (N/2)(N/2 - 1)(N/2 - 2)/6$ while $P_N < N(\ln N + \ln\ln N)$ (see [6, p. 69]) it follows from Lemma 2 that we can choose $s < P_N$ elements $d_i \in \mathscr{D}$ such that for $r_1 = r - d_1 - d_2 - \cdots - d_s$, $r_1 \equiv 0 \bmod P_N$. Further $r_1 \leqslant \Pi_N(1 - 1/(N-1))$. To prove this it suffices to show that $d_1 + d_2 + \cdots + d_s \leqslant \Pi_N(1 - 1/N) - \Pi_N(1 - 1/(N-1))$ since $r \geqslant \Pi_N(1 - 1/N)$. To see that this is so we note that $d_i \leqslant \Pi_{N-1}/p^3$ where $p = P_{[N/2]}$ while $s < P_N$. Thus $d_1 + \cdots + d_s < \Pi_{N-1}/p^3 \cdot P_N = \Pi_N/p^3$. Since [6, p. 69] $p = P_{[N/2]} > [N/2]\ln[N/2]$ we see that for large N, $p^3 > (N)(N-1)$. Thus $d_1 + \cdots + d_s \leqslant \Pi_N/p^3 < \Pi_N/N(N-1) = \Pi_N(1 - 1/N) - \Pi_N(1 - 1/(N-1))$. The claim is established.

If $r_1 \leqslant \Pi_N(2 - 1/(N-1))$, the process of Step I now stops.

If $r_1 > \Pi_N(2 - 1/(N-1))$ we proceed to subtract more elements of $\mathscr{D}$ from r_1 until it becomes sufficiently small; however this must be done in such a way that the result, say r_2, staisfies

$$1. \quad r' \equiv 0 \bmod P_N,$$

$$2. \quad \Pi_N(1 - 1/(N-1)) \leqslant r' \leqslant \Pi_N(2 - 1/(N-1)).$$

In order to assure that $r' \equiv 0 \bmod P_N$ we subtract off elements from $\mathscr{D}$, at most P_N at a time, such that the sum of the divisors subtracted is $\equiv 0 \bmod P_N$ and condition 1 will hold. Since the divisors are all less than Π_{N-1} and we are subtracting P_N at a time and the interval r' we wish in which to be has length $\Pi_N = \Pi_{N-1} \cdot P_N$, we can subtract in such a way as to end up in the desired interval, if the total of all available divisors, properly grouped, is large enough to bring the largest value of r_1 below $\Pi_N(2 - 1/(N-1))$. Since $r_1 \leqslant r \leqslant \Pi_N(2 - 1/N)$, we must show that the sum of the divisors is at least $\Pi_N(2 - 1/N) - \Pi_N(2 - 1/(N-1)) = \Pi_N/N(N-1)$. But we can continue to subtract groups of at most P_N divisors from $\mathscr{D}$ until there remains less than P_N elements. Thus of all the divisors in $\mathscr{D}$ we will be able, if needed, to subtract all but at most P_N of them. It follows that we may subtract at least

$$\left(\frac{(N/2)(N/2-1)(N/2-2)}{6} - P_N\right)$$

divisors each of which is at least as large as Π_{N-3}. For N sufficiently large the number of divisors is at least $N^3/100$, so that we are done if $\Pi_{N-3}(N^3/100) \geqslant \Pi_N/N(N-1)$ which is equivalent to

$$N^5 - N^4 \geqslant 100 P_N P_{N-1} P_{N-2}$$

which holds for N sufficiently large since $P_N < N(\ln N + \ln\ln N)$. Thus Step I can be completed.

We note that we have thus written $r = r_1 + d_1 + d_2 + \cdots + d_N$ where

1. $d_i \mid \Pi_{N-1}$, d_i distinct,
2. $d_i \geqslant \Pi_{N-3}$,
3. $r_1 \equiv 0 \bmod P_N$,
4. $\Pi_N(1 - 1/(N-1)) \leqslant r_1 \leqslant \Pi_N(2 - 1/(N-1))$.

Step II. Let $r_2 = r_1/P_N$. Then by conditions 3 and 4 we see that r_2 is an integer and

$$\Pi_{N-1}(1 - 1/(N-1)) \leqslant r_2 \leqslant \Pi_{N-1}(2 - 1/(N-1)).$$

Thus, by induction there are $d_i' \mid \Pi_{N-1}$, d_i' distinct, $d_i' \geqslant \Pi_{N-4}$ such that $r_2 = \sum d_i'$. Let $d_i'' = P_N d_i'$. Thus $d_i'' \mid \Pi_N$, $d'' \nmid \Pi_{N-1}$, so that the d_i'' are distinct both from each other and from the d_i choosen in Step I. Furthermore, $d_i'' \geqslant c\Pi_{N-4}P_N > c\Pi_{N-3}$. Also since $r = P_N r_2 + \sum d_i$ we see that

$$r = \sum d_i'' + \sum d_i$$

is an expansion which satisfies all the conditions of Lemma 2 for $k = N$. The lemma follows by induction.

LEMMA 4. *If* $\Pi_{k-1} \leqslant N \leqslant \Pi_k$ *then*

$$k \leqslant \frac{\ln N}{\ln\ln N}\left(1 + \frac{\ln\ln\ln N}{\ln\ln N}\right).$$

Proof. From [6, p. 70], we see that $\ln \Pi_k \geqslant P_k(1 - 1/2 \ln P_k)$. Thus an upper bound for k is the smallest integer k_0 such that $P_{k_0}(1 - 1/2 \ln P_{k_0}) > \ln N$ where $P_k \geqslant k(\ln k + \ln\ln k - 3/2)$. For k_0 equal to the bound given in the lemma this yields

$$\begin{aligned}\ln \Pi_{k_0} &\geqslant \left(1 - \frac{1}{2 \ln P_{k_0}}\right) k(\ln k_0 + \ln\ln k_0 - 3/2)\\ &= \left(1 - \frac{1}{2\ln P_N}\right)\frac{\ln N}{\ln\ln N}\left(1 + \frac{\ln\ln\ln N}{\ln\ln N}\right)\\ &\quad\times\left\{\ln\ln N + \ln\left(1 + \frac{\ln\ln\ln N}{\ln\ln N}\right)\right.\\ &\quad\left. + \ln\left(1 - \frac{\ln\ln\ln N}{\ln N} + \frac{\ln\left(1 + \frac{\ln\ln\ln N}{\ln\ln N}\right)}{\ln\ln N}\right) - 3/2\right\}\\ &\geqslant \frac{\ln N}{\ln\ln N}\left(1 + \frac{\ln\ln\ln N}{\ln\ln N}\right)\{\ln\ln N - 2\}.\end{aligned}$$

Since for large N the two middle logarithmic terms in the braces are both close to zero. Thus,

$$\ln \Pi_{k_0} \geqslant \ln N \left(1 + \frac{\ln\ln\ln N}{\ln\ln N}\right)\left(1 - \frac{2}{\ln\ln N}\right) > \ln N.$$

Thus for N large enough there is an integral value of k less than the given bound which would also satisfy

$$\ln \Pi_k > \ln N.$$

THEOREM 2. *There is a constant K so that for every $N \geqslant 2$, $D(N) \leqslant KN(\ln N)^3$.*

Proof. Given the fraction a/N in the unit interval we find k so that $\Pi_{k-1} < N \leqslant \Pi_k$. If $N \mid \Pi_k$ we rewrite $a/N = b/\Pi_k$ and by Lemma 1, $b = \sum d_i$, $d_i \mid \Pi_k$. This yields an Egyptian expansion of a/N with the largest denominator at most Π_k. Since $P_k < k(\ln k + \ln\ln k) < k^2$ and Lemma 4 gives a bound for k, we get that the denominators in this case are certainly less than $N(\ln N)^3$.

We next consider the case in which $N \nmid \Pi_k$. In this case

$$\frac{a}{N} = \frac{a\Pi_k}{N\Pi_k} = \frac{qN + r}{N\Pi_k} = \frac{q}{\Pi_k} + \frac{r}{N\Pi_k}$$

where r is chosen so that $\Pi_k(1-1/k) \leqslant r \leqslant \Pi_k(2-1/k)$. This can be done since we may assume $a \geqslant 2$ and since $N \leqslant \Pi_k$. The fraction q/Π_k can be handled as the case $N \mid \Pi_k$. We need only consider $r/N\Pi_k = (1/N)(r/\Pi_k)$. If we get an expansion for r/Π_k and multiply each denominator by N then since $N \nmid \Pi_k$, they will all be distinct from those used to expand q/Π_k. By Lemma 3 there are divisors d_i of Π_k such that

$$r = \sum d_i, \qquad d_i \geqslant c\Pi_{k-3}.$$

Thus the denominators in the expansion of r/Π_k are at most $c^{-1}P_kP_{k-1}P_{k-2}$. Thus the denominators in expansion of $r/N\Pi_k$ are at most $c^{-1}NP_kP_{k-1}P_{k-2}$. Using the upper bound in Lemma 4 for k one can show after some calculation that

$$c^{-1}NP_kP_{k-1}P_{k-2} \leqslant 2c^{-1}N(\ln N)^3.$$

Thus the theorem is established.

III. Some Special Cases and Numerical Results

Theorem 3. *$D(N) = N$ for $N = 2^n, \Pi_n$ or $n!$, $n = 1, 2, 3,\ldots$.*

Proof. For $a/2^n$ we write a as a sum of powers of 2 (base 2) and cancel to get an Egyptian expansion. For $N = \Pi_n$ we use Lemma 1. For $N = n!$ we use the analog of Lemma 1 with Π_n replaced by $n!$. Since this modified Lemma 1 is easy to prove, we omit the proof.

Theorem 4. *For $n = 1, 2, 3,\ldots,$ we have $D(3^n) = 2 \cdot 3^n$.*

Proof. Given $a/3^n$ we rewrite it as $2a/2 \cdot 3^n$ and expand $2a$ according to its base 3 expansion $2a = \sum_{i=0}^{n-1} \epsilon_i 3^i$ where $\epsilon_i = 0, 1,$ or 2 since each of the terms in the sum divides $2 \cdot 3^n$ we see $D(3) = 2 \cdot 3^n$. At least one denominator in the expansion of $2/3^n$ must be divisible by 3^n. If only one denominator is so divisible, and it is 3^n, then the remaining terms would be an expansion of $1/3^n$ in which no term is divisible by 3^n, a contradiction. Hence, $D(3^n) \geqslant 2 \cdot 3^n$.

Theorem 5. *For $N = P^n$, P a prime we get $D(P^n) \leqslant 2P^{n-1}D(P)$.*

We may restrict our attention to $P \geqslant 5$, since the preceding two theorems handle $P = 2$ and $P = 3$.

If $a/P^n > 1/2$ we consider $b/2P^n = a/P^n - 1/2$ where $b < P^n$ otherwise we consider $2a/2P^n = b/2P^n$ where again $b < P^n$. We next expand $b/2P^n$ in the Egyptian form with denominators at most $2P^{n-1}D(P)$, since

$b/P^n < 1/2$, $1/2$ will not be used and can be added on at the end if $a/P^n > 1/2$. We write $b = \sum_{i=0}^{n-1} \epsilon_i P^i$, $0 \leqslant \epsilon_i < P$. Thus

$$\frac{b}{P^n} = \sum_{i=0}^{n-1} \frac{\epsilon_i}{P^{n-i}} = \sum_{i=0}^{n-1} \frac{\epsilon_i}{P} \cdot \frac{1}{P^{n-i-1}}.$$

For each i, $0 \leqslant i \leqslant n-1$ we can expand $\epsilon_i/P = \sum_{j=1}^{k_i} 1/n_j^{(i)}$, $2 \leqslant n_i \leqslant D(P)$. Thus $b/P^n = \sum_{i=0}^{n-1} \sum_{j=1}^{k_i} 1/2n^{(i)}P^{n-i-1}$. A slight difficulty arises in that the denominators may not be distinct. However we know that for all P, $D(P) \leqslant P(P-1)$ (see [2, Theorem 3, p. 347]), thus the only equalities which can arise are of the form

$$\frac{1}{2n_{j_1}^{(i)}P^{n-i-1}} = \frac{1}{2n_{j_2}^{(i+1)}P^{n-i}}. \qquad (*)$$

So that $n_{j_1}^{(i)} = n_1 = Pn_2 = Pn_{j_2}^{(i+1)}$. Since $n_1 \leqslant P(P-1)$ we see that $n_2 \leqslant (P-1)$. In all instances where equalities like (*) occur we replace these two terms by the one term $1/n_2P^{n-i}$. If n_2 is odd it can not be equal to any other term. If n_2 is even it may be that $1/n_2P^{n-i}$ is equal to another term, which is of the form $1/2n_x^{(i)}P^{n-i-1}$ or $1/2n_y^{i-1}P^{n-i}$, but not both since otherwise these would have been reduced. Let $n_3 = n_y^{(i-1)}$. These two equal terms may be replaced by $1/n_3P^{n-i}$.

If n_3 is odd it is distinct from all other terms, since the only way $1/n_3P^{n-i}$ could have occured was if it came from the reduction of two terms at the previous stop, but in that case both $1/2n_x^{(i)}P^{n-i-1}$ and $1/2n_y^{(i-1)}P^{n-i}$ would have been replaced earlier, and $1/n_2P^{n-i}$ could not have equaled any other term. If n_3 is even possible new equalities may occur, but since $n_1 < P$ after at most $\log_2 P$ steps, this process must terminate yielding the desired expansion. The theorem is proved.

The last theorem of this section has to do with the nonunicity of Egyptian Fractions.

THEOREM 6*. *If $n_1 < n_2 < n_3 < \cdots$ is an infinite sequence of positive integers such that every rational number* (0, 1) *can be represented as*

$$\frac{a}{b} = \frac{1}{n_{i_1}} + \frac{1}{n_{i_2}} + \cdots + \frac{1}{n_{i_k}}$$

for some k and distinct n_{i_j} in the sequence. Then there is at least one rational number which has more than one representation.

* The authors would like to thank Drs. Graham and Lovasz for helpful discussions about this theorem.

Proof. Since $\sum_{i=1}^{\infty} 1/2^i = 1$ we see that for some value of i, $n_{i+1} < 2n_i$. Thus $1/n_i - 1/n_{i+1} < 1/n_{i+1}$. By the hypothesis $1/n_i - 1/n_{i+1} = \sum_{j=1}^{k} 1/n_{i_j}$. So that for $i_0 = i + 1$, $1/n_i = \sum_{j=0}^{k} 1/n_{i_j}$. But each side of this equation yields an acceptable expansion of $1/n_i$. Thus the theorem is proved.

We also note that either $1/n_i$ is used infinitely often or there is another subscript j such that $n_{j+1} < 2n_j$, which in turn is used infinitely often or there is another subscript l such that $n_{l+1} < 2n_l$, etc. Thus there are in fact infinitely many rationals with more than one representation. It is probably true that some fraction must have infinitely many representations.

We conclude this section with some numerical results. The following table gives an indication of what happens for the first few primes. A

N	$\{\{\log_2 N\}\}$	$D(N)/N$	occurrence of $D(N)$
2	1	1	$\frac{1}{2}$
3	2	2	$\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$
5	3	3	$\frac{2}{5} = \frac{1}{3} + \frac{1}{15}$
7	3	4	$\frac{2}{7} = \frac{1}{4} + \frac{1}{28}$
11	4	4	$\frac{2}{11} = \frac{1}{12} + \frac{1}{22} + \frac{1}{33} + \frac{1}{44}$
13	4	5	$\frac{2}{13} = \frac{1}{10} + \frac{1}{26} + \frac{1}{65}$
17	5	5	$\frac{4}{17} = \frac{1}{12} + \frac{1}{15} + \frac{1}{17} + \frac{1}{4 \cdot 17} + \frac{1}{5 \cdot 17}$
19	5	6	$\frac{2}{19} = \frac{1}{12} + \frac{1}{4 \cdot 19} + \frac{1}{6 \cdot 19}$
23	5	6	$\frac{2}{23} = \frac{1}{23} + \frac{1}{2 \cdot 23} + \frac{1}{3 \cdot 23} + \frac{1}{6 \cdot 23}$
29	5	6	$\frac{5}{29} = \frac{1}{6} + \frac{1}{6 \cdot 29}$
31	5	6	$\frac{4}{31} = \frac{1}{12} + \frac{1}{31} + \frac{1}{4 \cdot 31} + \frac{1}{6 \cdot 31}$
37	6	8	$\frac{12}{37} = \frac{1}{6} + \frac{1}{8} + \frac{1}{2 \cdot 37} + \frac{1}{3 \cdot 37} + \frac{1}{4 \cdot 37} + \frac{1}{8 \cdot 37}$

comparison of the second and third columns shows that the bound of Theorem 1 is frequently low.

We conclude with a numerical example which illustrates that whichever purpose one desires, minimizing k or n_k the algorithms to date leave something to be desired. We expand 5/121 by several algorithms.

The Fibonacci–Sylvester [7] algorithm yields

$$\frac{5}{121} = -\frac{1}{25} + \frac{1}{757} + \frac{1}{763308} + \frac{1}{873960180913} + \frac{1}{15\,276\,184\,876\,404\,402\,665\,313}.$$

The Erdös algorithm [3] yields considerably smaller denominators, but is longer:

$$\frac{5}{121} = \frac{1}{48} + \frac{1}{72} + \frac{1}{180} + \frac{1}{1452} + \frac{1}{4354} + \frac{1}{8712} + \frac{1}{87120}.$$

The continued fraction algorithm [2] yields

$$\frac{5}{121} = \frac{1}{25} + \frac{1}{1225} + \frac{1}{3477} + \frac{1}{7081} + \frac{1}{11737}.$$

The algorithm presented here in Theorem 2 yields:

$$\frac{5}{121} = \frac{5(2 \cdot 3 \cdot 5 \cdot 7)}{121 \cdot (2 \cdot 3 \cdot 5 \cdot 7)} = \frac{7 \cdot 121 + 203}{121 \cdot 2 \cdot 3 \cdot 5 \cdot 7}.$$

Since $203 = 7(3 \cdot 5 + 2 \cdot 5 + 3 + 1)$, this gives

$$\frac{5}{121} = \frac{1}{30} + \frac{1}{242} + \frac{1}{363} + \frac{1}{1210} + \frac{1}{3630},$$

which is considerably better.

However modifying our present algorithm in an ad hoc way yields the following two better expansions. We have

$$\frac{5}{121} = \frac{8 \cdot 121 + 82}{121 \cdot 2 \cdot 3 \cdot 5 \cdot 7}.$$

By replacing 82 by $77 + 5$ and 8 by $5 + 3$ we get a good short expansion, namely,

$$\frac{5}{121} = \frac{1}{42} + \frac{1}{70} + \frac{1}{330} + \frac{1}{5082},$$

while replacing 82 by 33 + 35 + 14 yields

$$\frac{5}{121} = \frac{1}{42} + \frac{1}{70} + \frac{1}{726} + \frac{1}{770} + \frac{1}{1815},$$

which while longer has denominators considerably smaller than any of the others.

IV. Some Conjectures

In working on these and related problems some conjectures arose which we are not yet able to prove.

Conjecture 1. *The constant in Lemma 2 can be replaced by* 1.

Numerical evidence for low values of k support this and of course since the induction doesn't change the constant, a finite but difficult computation can settle this. Hopefully a clever trick can do it more easily.

An affirmative answer to this conjecture implies the constant in Theorem 2 can also be taken to be 1.

Conjecture 2. *$D(N)$ is submultiplicative, i.e., $D(N \cdot M) \leqslant D(N) \cdot D(M)$. If true, relative primeness of M and N is probably irrelevant.*

This would enable one to concentrate on $N = P$ in proving bounds for $D(N)$. One might note that instead of splitting cases on $N \mid \Pi_k$, $N \nmid \Pi_k$ we could in general use denominator $N'\Pi_k$ when $N' = N/d$, $d = (N, \Pi_k)$, to get a more efficient method of expanding a/N with small denominators.

Conjecture 3. *For every $\epsilon > 0$ there is a constant $K = K(\epsilon)$ such that $D(N) \leqslant KN(\ln N)^{1+\epsilon}$.*

Conjecture 4. *Let $n_1 < n_2 < \cdots$ be an infinite sequence of positive integers such that $n_{i+1}/n_i > c > 1$. Can the set of rationals a/b for which*

$$\frac{a}{b} = \frac{1}{n_{i_1}} + \frac{1}{n_{i_2}} + \cdots + \frac{1}{n_{i_t}}$$

is solvable for some t contain all the rationals in some interval (α, β). We conjecture not.

If this conjecture is true then according to Graham [5] this is best possible.

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