ON THE IRRATIONALITY OF CERTAIN SERIES

Par P. Erdös

During my long life I spent lots of time thinking about irrationality of series. The reader with a little maliciousness may say : " spent and wasted" since I neither discovered any new general methods nor had any spectacular success like Apery. Nevertheless I hope to convince the reader that not all of it was completely wasted. First of all I state some of my previous results several of which were obtained with E. Straus. I state many unsolved problems and prove also some new theorems.

I proved more than 30 years ago [1] that for every integer t, $\sum_{n=1}^{\infty} \frac{d(n)}{t^n} = \sum_{n=1}^{\infty} \frac{1}{t^n-1}$ is irrational. Chowla conjectured that the same holds for rational t, this is almost certainly true, but is unattackable by my methods. I can also prove that $\sum_{n=1}^{\infty} \frac{d(n)}{2^n+1}$ is irrational, but can do nothing with say $\sum_{n=1}^{\infty} \frac{d(n)}{2^n-3}$ or $\sum_{n=1}^{\infty} \frac{\nu(n)}{2^n}$ ($\nu(n)$ denotes the number of distinct prime factors of n, $d(n)$ denotes the number of divisors, $\phi(n)$ Euler's ϕ function and $\sigma_k(n)$ the sum of the k-th powers of the divisors of n). Probably $\sum_{n=1}^{\infty} \frac{\phi(n)}{2^n}$ and $\sum_{n=1}^{\infty} \frac{\sigma(n)}{2^n}$ are also irrational but this is not attackable by my methods. I found it particularly annoying that I could not prove the irrationality of $\sum_{n=2}^{\infty} \frac{1}{n!-1}$.

Kac and I [2] proved that $\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n!}$ is irrational for $k=1$ and $k=2$ but our proof does not seem to work for $k > 2$ though of course the result remains true.

Straus and I [3] proved that if $1 < a_1 < \ldots$ is a sequence of integers then $\sum_{n=1}^{\infty} \frac{d(n)}{a_1 \ldots a_n}$ is irrational.

We conjectured that it suffices to assume that $a_n \to \infty$ but could never prove it.

We also conjectured that if $a_{n+1} \geq a_n$ and $a_n \to \infty$ then

(1) $$\sum_{n=1}^{\infty} \frac{\phi(n)}{a_1 \cdots a_n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\sigma(n)}{a_1 \cdots a_n}$$

are both irrational, but we could only prove (1) if we make some further assumptions on the growth properties of the a's. Observe that $a_n = \phi(n)+1$ and $a_n = \sigma(n)+1$ shows that $a_n \to \infty$ does not suffice for the irrationality of (1).

I further proved that $\sum_{n=1}^{\infty} \frac{p_n^k}{n!}$, $p_1 < p_2 < \ldots$ is the sequence of primes, is irrational for every k. I could not prove that $\sum_{n=1}^{\infty} \frac{p_n^k}{2^n}$ is also irrational (this seems very difficult already for $k=1$). Let $2 \leq g_1 \leq g_2 \leq \ldots$ be any sequence of integers it seems reasonable to expect that if $g_n = o(p_n)$ then

(2) $$\sum_{n=1}^{\infty} \frac{p_n}{g_1 \cdots g_n}$$

is irrational. In fact probably $g_n \geq 2$, $g_n = o(p_n)$ suffices for the irrationality of (2). I can only prove the irrationality of (2) under very much more restrictive conditions. $g_n = p_n+1$ again shows that some condition is indeed needed for the irrationality of (2). It is not hard (but not entirely trivial) to prove that $g_n > g_{n-1}$ and $g_n = o(p_n)$ then (2) is irrational.

I further observed that if $\{n_k\}$ is any sequence of integers satisfying $n_k^{1/2^k} \to \infty$ then $\sum \frac{1}{n_k}$ is irrational. It is easy to see that this result is best possible, since $n_k > c^{2^k}$ does not suffice. On the other hand I proved [5] that if $n_k > k^{1+\varepsilon}$ holds for all k and $\limsup n_k^{1/2^k} = \infty$ then $\sum \frac{1}{n_k}$ is irrational. My proof is not entirely trivial. It seems possible that other theorems of this kind can be proved but I dit not investigate this possibility carefully so far.

Once I asked : Assume that $\sum_{k=1}^{\infty} \frac{1}{n_k}$ and $\sum_{k=1}^{\infty} \frac{1}{n-1}$ are both rational. How fast can n_k tend to infinity ? I was (and am) sure that $n_k^{1/k} \to \infty$ is possible, but $n_k > (1+c)^{2^k}$ is not possible. Unfortunately almost nothing is known. David Cantor observed that

$$\sum_{k=3}^{\infty} \frac{1}{\binom{k}{2}} \quad \text{and} \quad \sum_{k=3}^{\infty} \frac{1}{\binom{k}{2}-1}$$

are both rational and we do not know any sequence n_k with this property

which tends to infinity faster. Stolarsky asked the following beautiful question : Let $\sum \frac{1}{n_k} < \infty$. Is it true that there always is an integer t for which $\sum_{n_k \neq t} \frac{1}{n_k - t}$ is irrational ? So far we could make no progress with this pretty question.

Straus and I proved that the set of points in the plane of the form

$$X = \sum \frac{1}{n_k}, \quad y = \sum \frac{1}{n_k - 1}$$

contains open sets. We did not publish our proof since we hope to generalise it to r dimensions.

Strengthening previous results of Straus and myself I proved [6] that if $\limsup_{k \to \infty} n_k / k^{\ell} = \infty$ then $\sum_{k=1}^{\infty} \frac{1}{2^{n_k}}$ can not satisfy an algebraic equation of degree $\leq \ell$ having integer coefficients. Perhaps $\limsup_{k \to \infty} n_k / k = \infty$ suffices, but I could not even prove that if $n_k > ck^2$ then $\left(\sum_{k=1}^{\infty} \frac{1}{2^{n_k}} \right)^2$ is always irrational. Perhaps here I overlook a simple idea.

In [5] I define a sequence $n_1 < n_2 < \ldots$ to be an irrationality sequence if for every set of integers $c_i > 0$ $\sum_{i=1}^{\infty} \frac{1}{c_i n_i}$ is irrational e.g it is easy to see that $n_k = k!$ is not an irrationality sequence. I prove in [5] that $n_k = 2^{2^k}$ is an irrationality sequence. The proof is not entirely trivial.

In the present paper I prove the following

Theorem. - *Let $a_1 < a_2 < \ldots$ be a sequence of integers satisfying $a_{n+1} - a_n \to \infty$. Then $\sum_{n=1}^{\infty} \frac{a_n}{2^{a_n}}$ is irrational.*

I conjectured this more than 20 years ago but at that time only a much weaker result was proved [7]. The theorem of course remains true if 2 is replaced by any other integer $t > 1$.

It seems very likely that $\lim_{n \to \infty} a_n / n = \infty$ suffices for the irrationality of $\sum_{n=1}^{\infty} a_n / 2^{a_n}$. In fact I could not find an example of a sequence $\{a_n\}$ for which $\limsup_{n \to \infty} (a_{n+1} - a_n) = \infty$ and $\sum_{n=1}^{\infty} a_n / 2^{a_n}$ is rational, but here I think that such an example exists and in fact might be easy to find and I probably just overlook an obvious point.

Now we prove our theorem. The proof will not be entirely trivial but as stated in the introduction it will not use any really new idea. Let us assume that our theorem is false. Then we would have

$$(3)\qquad \sum_{n=1}^{\infty} \frac{a_n}{2^{a_n}} = \frac{u}{v\,2^r}\,, \quad r \geq 0, \quad v \equiv 1 \pmod 2, \quad (u, v\,2^r) = 1.$$

We will now show that (3) leads to a contradiction. Let k be large and let $a_n = 2^k - s$ be the largest $a_n < 2^k$, $a_{n+1} = 2^k + t_1$, $a_{n+2} = 2^k + t_2$, ... We really should write $a_n = 2^k - s^{(k)}$, $a_{n+1} = 2^k + t_1^{(k)}$ ... but where there is no danger of confusion we will omit the upper index k. Multiply both sides of (3) by $v\,2^{2^k - s}$. Clearly (for sufficiently large k) the left side becomes an integer. Thus the right side must become an integer too. Thus

$$(4)\qquad v\left(\frac{2^{k+t_1}}{2^{t_1+s}} + \frac{2^{k+t_2}}{2^{t_2+s}} + \ldots\right)$$

must be an integer and therefore must be ≥ 1.

Assume first

$$(5)\qquad t_1 + s > k.$$

Observe that if (5) holds then since v is odd the fractional part of $\frac{v\,2^k}{2^{t_1+s}}$ is positive, but is at most $1 - \frac{1}{v}$. Thus since the sum is ≥ 1 we must have (by (4))

$$(6)\qquad \frac{t_1}{2^{t_1+s}} + \frac{2^{k+t_2}}{2^{t_2+s}} + \ldots \geq \frac{1}{v^2}\,.$$

(6) easily leads to a contradiction since as we now show the left side of (6) tends to 0 as k tends to infinity. First observe that (5) immediately implies that

$$(7)\qquad \frac{t_1}{2^{t_1+s}} \to 0 \quad \text{as} \quad k \to \infty$$

Further

$$(8)\qquad \frac{2^{k+t_2}}{2^{t_2+s}} + \frac{2^{k+t_3}}{2^{t_3+s}} + \ldots \to 0$$

immediately follows from (5) and $t_{i+1} - t_i \to \infty$ (this follows immediately from $a_{n+1} - a_n \to \infty$).

(7) and (8) implies that the left side of (6) tends to 0 and hence (6) can not hold. Thus if (5) holds our theorem is proved.

Thus we can assume that (5) is false and hence $t_1 + s \leq k$. Let i be defined by

$$(9)\qquad t_1 + s \leq k < t_{i+1} + s.$$

By (9), (4) can be written as (I_1, I_2 and I_3 are integers)

$$(10)\qquad \begin{cases} I_1 = I_2 + v \sum_{j=1}^{i} \frac{t_j}{2^{t_j+s}} + v \sum_{j=1}^{\infty} \frac{2^{k+t_{i+j}}}{2^{t_{i+1}+s}} = I_2 + v\left(\frac{t_1}{2^{t_1+s}} + \frac{2^{k+t_{i+1}}}{2^{t_{i+1}+s}}\right) + \\ v\left(\sum_{j=2}^{i} \frac{t_j}{2^{t_j+s}} + \sum_{j=2}^{\infty} \frac{2^{k+t_{i+j}}}{2^{t_{i+j}+s}}\right) = I_2 + \Sigma_1 + \Sigma_2 + \Sigma_3\,. \end{cases}$$

Observe that Σ_3 is an infinite sum and since our sequence $a_1 < \dots$ was infinite $\Sigma_3 > 0$. Thus from (10)

$$\Sigma_1 + \Sigma_2 + \Sigma_3 = I_3 \geq 1. \tag{11}$$

Σ_2 might be empty but if it is not empty then it tends to 0 as k tends to infinity. This follows immediately from $t_{u+1}-t_u \to \infty$. Σ_3 also tends to 0 as k tends to infinity, this again follows immediately from $t_{u+1}-t_u \to \infty$ and $t_{i+1}+s \geq k$. Thus (11) can hold only if Σ_1 is not an integer but its fractional part tends to 1. This case needs a little more analysis and is the only difficult part of our proof.

First of all observe that

$$\frac{v\, t_{i+1}}{2^{t_{i+1}+s}} \to 0 \quad \text{as} \quad k \to \infty. \tag{12}$$

(12) immediately follows from $t_{i+1}+s > k$. The remaining part of Σ_1 is

$$v\left(\frac{t_1}{2^{t_1+s}} + \frac{2^k}{2^{t_{i+1}+s}}\right) = \Sigma. \tag{13}$$

If Σ is an integer we arrived at our contradiction. Thus assume that Σ is not an integer. To obtain our contradiction it suffices to show that

$$\Sigma - (\Sigma) < 1-\delta \tag{14}$$

where $\delta > 0$ an absolute constant which does not depend on k (clearly (14) contradicts (11)). Write

$$\frac{v\, 2^k}{2^{t_{i+1}+s}} = \frac{v}{2^u}, \quad u > 0 \quad \text{since} \quad t_{i+1}+s > k.$$

Thus to prove (14) we have to show that

$$v\left(\frac{t_1}{2^{t_1+s}} + \frac{1}{2^u}\right) - \left[v\left(\frac{t_1}{2^{t_1+s}} + \frac{1}{2^u}\right)\right] < 1-\delta. \tag{15}$$

The proof of (15) is easy. The fractional part of $\frac{v}{2^u}$ is at most $\frac{1}{2}$ and a simple computation shows that the fractional part of $\frac{v\, t_1}{2^{t_1+s}}$ is less than $1-2^{-(v+10)}$. In fact much more is true. The reason for this crude inequality is that if $t_1+s \leq v+10$ then the inequality is obvious and if $t_1+s \geq v+10$ then it is easy to see that $v\, t_1 < \frac{1}{2}\, 2^{t_1+s}$ and the fractional part is in fact $< \frac{1}{2}$. Put

$$\max\,(t_1+s, u) = L.$$

If $L \leq v+20$ then the fractional part of (15) is $\leq 1 - \frac{1}{2^{v+20}}$, which completes the proof of (15). If on the other hand $L > 2v+20$, then either $u > 2v+20$ but then by what has been just proved the fractional part of (15) is less than

$$\frac{v}{2^{2v+20}} + 1-2^{-v-10} < 1-2^{-v-11}$$

and (15) is again proved, or $t_1+s > 2v+20$, but then it is easy to see that $\frac{v\, t_1}{2^{t_1+s}} < \frac{1}{4}$ and hence the fractional part of (15) is less than $\frac{3}{4}$, which completes the proof of (15) and our theorem. Perhaps our proof of (15) is much too complicated.

Our condition $a_{n+1}-a_n \to \infty$ could be weakened somewhat but at the moment I can not replace it by an interesting weaker condition.

Let α be any rational number less than 1. Write $\alpha = \sum_{n=1}^{\infty} \frac{a_n}{2^{a_n}}$ where the a_n are determined by the "greedy algorithm" more precisely : if $a_1, \ldots, a_n$ is already defined I choose the largest possible value for a_{n+1}. Is there an α (every α) for which $a_{n+1}-a_n$ is unbounded ? Perhaps I overlook a trivial point but I made no progress with this question.

Is it true that for every $c > 0$ there is a sequence of rational (real) numbers $\{u_n\}$ $u_n > (1+c)\, u_{n+1}$ so that if $\sum_{i=1}^{\infty} u_{n_i}$ is rational then $n_{i+1}-n_i$ is bounded.

Several times I tried to prove that $\sum_{n=1}^{\infty} \frac{g_n}{2^{g_n}}$ is rational where $g_1 < \ldots$ is the sequence of squarefree numbers. Perhaps I missed a simple idea but in any case I was completely unsuccessful.

REFERENCES

[1] P. Erdös. - *An arithmetic properties of Lambert series,* Indian J. Math. 12 p. 63-66 (1948).

[2] P. Erdös and M. Kac. - Amer. Math. Monthly, Problem 4518, p. 264, vol. 61 (1964).

[3] P. Erdös and E. Straus. - *Some number theoretic results,* Pacific J. Math. 36 p. 635-646 (1971), and *An the irrationality of certain series ibid.*

[4] P. Erdös. - *Sur certaines series à valeur irrationnelle,* Enseignement Math 4 p. 93-100 (1958).

[5] P. Erdös. - *Some problems and results on the irrationality of the sum of infinite series,* J. Math. Sci, 10, p. 1-7 (1975).

[6] P. Erdös. - *On the irrationality of certain series,* Indag. Math. 19 p. 212-219 (1957). Several other results of a somewhat special character are proved in this paper.

Paul Erdös,
Akademia Matematikai Intezete,
Realtanoda u. 13-15,
H-1053 Budapest, Hongrie
et

Département de Mathématiques,
U.E.R. des Sciences de Limoges,
123, rue Albert-Thomas,
87100 Limoges.