# SOME RECENT PROGRESS ON EXTREMAL PROBLEMS IN GRAPH THEORY

P. Erdös

Several survey papers have recently been published on problems and results concerning extremal graph theory; at the end of the introduction I give a short list of some of these papers. In this paper, I discuss some special problems which interested me in the last few years and where some progress has been made towards the final solution. I will also restate a few older problems which perhaps were neglected but which seem interesting and are perhaps not hopeless.

I. P. Turan, Egy gráfelméleti nélsöérték feladatról, Mat. Fiz. Lapok 48 (1941), 436-452, see also On the Theory of graphs, Coll. Math. 3 (1954) 19-30.

II. P. Erdös, Extremal problems in graph theory, Theory of Graphs and its Applications (M. Fiedler, ed.), Acad. Press, New York, 1969, 29-36.

III. P. Erdös, Some recent results on extremal problems in graph theory, Theory of Graphs, Internat. Sympos. Rome, 1966, Gordon and Breach, New York, 1917, 117-130.

IV. P. Erdös, On some new inequalities concerning extremal properties of graphs, Theory of Graphs, Proc. Coll. Tihany, Hungary, 1966, 77-81.

V. M. Simonovits, A method for solving extremal graph problems in graph theory, stability problems, Ibid. 279-319.

VI. M. Simonovits, Extremal graph problems with conditions, Combinatorial theory and its applications, Proc. Coll. Soc. J. Bólyai, 4 (1969), 999-1012.

B. Bollobas is writing a comprehensive book on extremal problems in graph theory.

1. Denote by $G(k;l)$ a graph of $k$ vertices and $l$ edges. $f(n;G)$ is the smallest integer for which every $G_1(n;f(n;G))$ contains $G$ as a subgraph. $C_k$ denotes a circuit of $k$ edges. In this paragraph I discuss $f(n;C_4)$.

First, a few historical and personal remarks. As is well known, the theory of extremal graphs really started when Turán determined $f(n;k_t)$ ($k_t$ is the complete graph of $k$ vertices) and raised several problems which showed the way to further progress. In 1936 I needed (the $c$'s will denote positive absolute constants)

$$(1) \qquad f(n;C_4) < c_1 n^{3/2}$$

for the following number theoretic problem. Denote by $h(x)$ the smallest integer so that if $1 \le a_1 < \dots < a_k \le x$, $k = h(x)$ is any sequence of integers, then the products $a_i a_j$ cannot all be distinct. I proved (1) without much difficulty and eventually deduced ($\pi(x)$ denotes the number of primes $\le x$)

$$\pi(x) + c_2 x^{3/4} / (\log x)^{3/2} < h(x) < \pi(x) + c_3 x^{3/4} / (\log x)^{3/2} .$$

I asked if (1) is best possible and Miss E. Klein proved

$$(2) \qquad f(n;C_4) > c_2 x^{3/2}$$

for every $c_2 > \frac{1}{2^{3/2}}$ and $n > n_0(c_2)$. Being struck by a curious blindness and lack of imagination, I did not at that time extend the problem from $C_4$ to other graphs and thus missed founding an interesting and fruitful new branch of graph theory. There is another curious fact about the prehistory of this subject. After Turán finished his paper I, he was informed by Mr. Krausz that W. Mantel and W.A. Wythoff proved (Wishundige Ungaven 10 (1907), 60-61) that every $G(n;[\frac{n^2}{4}]+1)$ contains a triangle. It seems certainly strange why they missed the obvious generalizations.

W. Brown and Rényi, V.T. Sós and I proved that

$$f(n;C_4) = (\tfrac{1}{2} + o(1))n^{3/2} . \tag{3}$$

Let $p$ be a prime or power of a prime. We in fact proved

$$f(p^2+p+1;C_4) \geq \tfrac{1}{2}(p^3+p)+p^2+1 \tag{4}$$

and conjecture that there is equality in (3).

The best upper bound I can get for $f(n;C_4)$ states:

$$f(n;C_4) \leq \tfrac{1}{2}n^{3/2} + n/4 - (3/16 + o(1))n^{1/2} . \tag{5}$$

The proof of (5) is not difficult. Let $G(n;l)$ be a graph not containing a $C_4$. Let $v(x_i)$, $i = 1,\dots,n$ be the valencies (or degrees) of the virtues of our graph. Since $G(n;l)$ contains no $C_4$ we must have

$$\sum_{i=1}^{n} \binom{v(x_i)}{2} \leq \binom{n}{2} . \tag{6}$$

To prove (6) observe that if (6) does not hold there are two vertices, say $x_1$ and $x_2$, both joined to two other vertices, say $x_3$ and $x_4$, i.e. $G(n;l)$ contains a $C_4$ which is impossible.

From $\sum_{i=1}^{n} v(x_i) = 2l$ and (6) we obtain by an elementary inequality

$$n\binom{\frac{2l}{n}}{2} \leq \sum_{i=1}^{n} \binom{v(x_i)}{2} \leq \binom{n}{2}$$

or

$$\frac{2l}{n}\left(\frac{2l}{n} - 1\right) \le n-1 \tag{7}$$

and (7) easily implies (5).

(5) can probably be improved. If there is equality in (4) we would expect that for every $n$

$$f(n;C_4) \le \frac{1}{2}n^{3/2} + \frac{n}{4} - \left(\frac{11}{16} + o(1)\right)n^{1/2}. \tag{8}$$

There are two possibilities for improving (5). By the friendship theorem we can not have equality in (6) and perhaps in (6), $\binom{n}{2}$ can in fact be replaced by $\binom{n}{2} - cn$ if we only consider graphs $G(n;l)$ with $v(x_i) < c'n^{1/2}$. It is in fact easy to see that for our purpose it suffices to consider such graphs, but I have not been able to make any progress here. Further, observe that $\frac{2l}{n}$ in general is not an integer and in any case the extreme graph does not have to be regular, but here too I got nowhere.

Using well known results on the distribution of primes, (4) and (5) gives

$$f(n;C_4) = \frac{1}{2}n^{3/2} + O\left(n^{3/2-c}\right),$$

but even if we assume $p_{n+1} - p_n = O(n^{\varepsilon})$ we only could get an error term $O(n^{1+\varepsilon})$.

Reimann constructs a bipartite graph of $n$ vertices and $(1+o(1))n^{3/2}/2\sqrt{2}$ edges which contains no $C_4$. He shows that if $n = 2(q^2+q+1)$ then his graph is in fact extremal.

Assume that $G(n)$ contains no $C_4$ and no $C_3$. Then perhaps ($e(G)$ is the number of edges of $G$).

$$\max e(G(n)) = \left(\frac{1}{2\sqrt{2}} + o(1)\right)n^{3/2}.$$

In a recent paper Bondy and Simonovits make a penetrating study of the $G(n)$ which contain no $C_{2k}$, but many unsolved problems remain.


P. Erdős, On sequences of integers no one of which divides the product of two others and on some related problems, Isv. Nauk. Inst. Math. Mech. Tomsh. 2(1938), 74-82; see also On some applications of graph theory to number theoretic problems, Publ. Ramanujan Inst. 1(1969), 131-136.

W.G. Brown, On graphs that do not contain a Thomsen graph, Canad. Math. Bull. 9(1966), 281-285; see also P. Erdős, A. Rényi and V.T. Sós, On a problem of graph theory, Studia Sci. Math. Hung. 1(1966), 215-235.

I. Reimann, Über ein Problem von K. Zarankievicz, Acta Math. Acad. Sci. Hung. 9(1958), 269-278.

A. Bondy and I. Simonovits, Cycles of even length in graphs, J. Combinatorial Theory 16(1974), 97-105.

2. Denote by $g$ the graph determined by the edges of a cube. Simonovits and I proved that

$$f(n;g) < cn^{8/5}. \tag{1}$$

It would be very interesting to decide if the exponent $^8/_5$ in (1) is best possible.

A theorem of Kőnári, the Turáns, and myself states that ($k(r,s)$ denotes the complete bipartite graph of $r$ white and $s$ black vertices)

$$f(n;k(r,r)) < c'_r n^{2-1/r} \tag{2}$$

W. Brown proved

$$f(n;k(3,3)) > c''_3 n^{5/3}.$$

It would be very desirable to prove that the exponent in (2) is best possible for every $r$ and in fact to prove that

$$f(n;k(r,r)) = (c_r+o(1))n^{2-1/r}. \tag{3}$$

As stated previously $c_2 = \frac{1}{2}$ but nothing is known for $r > 2$.

Simonovits and I conjectured that for every bipartite graph $G$ there is an $\alpha_G$, $0 < \alpha_G < 2$ so that

$$f(n;G)\Big/n^{1+\alpha_G} \to c_G, \quad 0 < c_G < \infty. \tag{4}$$

At first we thought that $\alpha_G$ must be either $\frac{1}{r}$ or $2 - \frac{1}{r}$, $r = 2,3,\ldots$, but we disproved this conjecture and now we believe that $\alpha_G$ is always rational and to every rational $\alpha$, $0 < \alpha < 1$ there is a $G$ with $\alpha_G = \alpha$.

(4) certainly no longer holds for hypergraphs. Denote by $G^{(r)}(k,l)$ an $r$-graph of $k$ vertices and $l$ $r$-tuples. W. Brown, V.T. Sós and I conjectured that

$$(5) \qquad f(n;G(6,3)) = o(n)^2$$

and in fact that

$$(6) \qquad f(n;G(6,3)) < n^{2-\varepsilon}$$

Szemerédi proved (5) and Ruzsa disproved (6), thus (4) does not hold for $r$-graphs. I recently conjectured that

$$f(n;G(k,k-3)) = o(n^2)$$

holds for every fixed $k$ if $n \to \infty$.

P. Erdős and M. Simonovits, Some extremal problems in graph theory, Comb. theory and its applications, Coll. Math. Soc. J. Bólyai Balatonfüred Hungary 377-390; see also The Art of Counting, Selected writings, P. Erdős, M.I.T. Press, 1973, 246-259.

T. Kövári, V.T. Sós and P. Turán, On a problem of K. Zarankievicz, Colloq. Math. 3, (1954), 50-57.

The paper of Ruzsa and Szemerédi will appear in Discrete Mathematics.

W. Brown, P. Erdős and V.T. Sós, On the existence of triangulated spheres in 3-graphs and related problems, Periodica Math. 3(1973), p. 227-228; see also Some extremal problems on r-graphs, in New Directions in the Theory of Graphs, ed. Frank Harary, Academic Press, 1973.

Some further extremal problems are stated in P. Erdős, Some unsolved problems in graph theory and combinatorial analysis, Combinatorial mathematics and its applications, Proc. Conference Oxford 1969 (Ed. D.J.A. Welsh), Acad. Press 1971, 97-109, see pp. 102-104.

3. Sauer and I asked the following question: Denote by $f(n,k)$ the smallest integer so that every $G(n;f(n,k))$ contains a regular graph of valency $k$ as a subgraph. Trivially $f(n,2) = n$ and it was a great surprise to us that we could get no satisfactory estimation even for $f(n,3)$. Our best upper bound is $f(n,3) < cn^{8/5}$ which follows from (1) of the previous chapter. Chvatal observed that $f(2n+3) > 6n$. His graph is defined as follows: Let the vertices of a $C_{2n}$ be $(x_1,\dots,x_{2n})$, $y_1$ is joined to all the $x_{2k+1}$ and $y_2$ to the $x_{2k}$, $k = 1,\dots,n$. $y$ is joined to all the $x$'s. This is our best lower bound!

One of the difficulties of the problem may be that there are too many regular graphs of valency three, and it is therefore difficult to consider the class of all of them. Perhaps the following question is simpler: Denote by $A(n)$ the smallest integer for which every $G(n;A(n))$ contains for some $k$ a $C_{2k}$ where $x_i$ and $x_{1+k}$ are joined by an edge for every $i = 1,\dots,k$. Clearly $f(n,3) \geq A(n)$ and $A(n) < cn^{5/3}$ since $K(3,3)$ is one of our graphs (for $k = 3$). We have no satisfactory upper or lower bound for $A(n)$. I expect $A(n) < n^{1+\varepsilon}$ for every $\varepsilon > 0$ and $n > n_0(\varepsilon)$, but perhaps $A(n)/n \to \infty$.

An older conjecture of Sauer and Berge states that every regular graph of valency four contains a regular subgraph of valency three. Chvatal just stated the following more general conjecture: Let g be a graph every vertex of which has valency $\geq 4$. Then g contains a regular subgraph of valency three.

Szemerédi recently posed the following problem: Denote by $F(n,k)$ the smallest integer for which every $G(n;F(n,k))$ contains a spanned regular subgraph of valency $k$. Clearly $F(n,k) \geq f(n,k)$. We have no satisfactory lower bound for $F(n,k)$ and know nothing better than Chvatal's $F(2n+3,3) \geq f(2n+3,3) \geq 6n$.

I proved $F(n,3) < c_1 n^{5/3}$. More precisely I showed: There is an absolute constant $c_1$ so that every $G(n;[c_1 n^{5/3}])$ either contains a $K_4$ or a spanned $K(3,3)$ (i.e. $G$ contains a graph of 6 vertices $x_1,x_2,x_3$; $y_1,y_2,y_3$ where $x_i$ is joined to $x_j$, $1 \leq i, j \leq 3$ but no two $x$'s or $y$'s are joined in $G$).

To prove our theorem first observe that by using Theorem 1 of our paper with Simonovits quoted in 2. it follows that without loss of generality we can assume that our graph has a subgraph $G(m)$ of $m$ vertices, $m > c_2 n^{2/15}$, each vertex of which has valency $> c_3 m^{2/3}$ where $c_3 = c_3(c_1)$ is large if $c_1$ is large.

If $G(m)$ contains a $K_4$ our theorem is proved. If it does not contain a $K_4$ then by a theorem of Szekeres and myself it contains an independent set of $[m^{1/3}]$ points $x_1,\ldots,x_l$, $l = [m^{1/3}]$. Let $x_{l+1},\ldots,x_m$ be the other vertices of our $G(m)$. Denote by $v(x_i)$ the valency of $x_i$

in $G(m)$, as stated $v(x_i) > c_3 m^{2/3}$. Thus each $x_i$, $1 \le i \le l$ is joined to more than $c_3 m^{2/3}$ $x_j$'s, $l < j \le m$. For sufficiently large $c_3$ we obtain by a simple computation

(1) $$\sum_{i=1}^{l} \binom{v(x_i)}{3} > l \binom{[c_3 m^{2/3}]}{3} > 5\binom{m}{3}.$$

Thus from (1) we obtain that there are three $x_i$'s, say $x_1$, $x_2$, $x_3$ which are joined to $x_{l+j}$, $1 \le j \le 6$. The graph spanned by $x_{l+j}$, $1 \le j \le 6$ cannot contain a triangle since then $x_1$ and this triangle would be a $K_4'$. Thus it contains an independent triangle, say $x_{l+1}$, $x_{l+2}$, $x_{l+3}$. But then since $x_1$, $x_2$, $x_3$ are independent by assumption $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ span a $K(3,3)$ in $G(m)$ and in $G(n)$, which completes our proof.

P. Erdős and G. Szekeres, A combinatorial problem in geometry, Compositio Math. 2(1935), 463-470.

4. In this final chapter I state some recent extremal problems and results on somewhat unconventional problems. Let $G(rn)$ be an $r$-partite graph having $n$ vertices of each color. Bollobás, Szemerédi and I conjectured that if each vertex has valency $\ge (r - \frac{3}{2})n$ then our graph contains a $K(r)$. We know that $r - \frac{3}{2}$ cannot be replaced by $r - \frac{3}{2} - \varepsilon$ but we cannot prove it even if $r - \frac{3}{2}$ is replaced by $r - 1 - \varepsilon$. Our paper on this and related questions will appear in Discrete Mathematics.

Let $G(n;e)$ be a graph of $n$ vertices and $e$ edges. Bollobás and I conjecture that if $e \geq \frac{n^2}{3}$ then our graph contains a triangle $\{x_1, x_2, x_3\}$ with

$$v(x_1) + v(x_2) + v(x_3) \geq \frac{6e}{n} \geq 2n. \tag{1}$$

We showed that (1) does not hold for $e < \frac{n^2}{3}$. We observed that every $G(n;e)$ has an edge $(x_1, x_2)$ with

$$v(x_1) + v(x_2) \geq \frac{4e}{n} . \tag{2}$$

(2) follows by a simple averaging process. It seems impossible to prove (1) by the same method.

I proved that every $G(n;[\frac{n^2}{4}] + 1$ has an edge, say $(x_1, x_2)$ with $\geq cn$ other vertices which are joined to both $x_1$ and $x_2$, i.e. the edge $(x_1, x_2)$ is on at least $cn$ triangles. Bollobás and I observed that $c \leq \frac{1}{6}$ and we could not decide whether $c = \frac{n}{6}$ .

Nordhaus and Stewart conjectured that every $G(n;[n^2/4]+k)$ contains at least $\frac{4nk}{9}$ triangles. Bollobás recently proved this conjecture.

Pósa proved that for $n \geq 4$, every $G(n;2n-3)$ contains a circuit with a diagonal and observed that $2n-3$ is best possible.

Denote by $r(n;k)$ the smallest integer for which every $G(n;r(n;k))$ has a circuit $C_l$ with a vertex which has at least $k-1$ diagonals (i.e. which is joined to at least $k+1$ other vertices of our $C_l$). The problem makes sense only for $n \geq k+2$ and it is easy to see that

$r(k+2,k) = \binom{k+2}{2} - \left[\frac{k+3}{2}\right]$. I conjectured that for $n > n_0(k)$ $r(n;k) = k(n-k) + 1$. The bipartite graph of $k$ white and $n-k$ black vertices shows that this conjecture, if true, is best possible. First I thought that $n_0(k) = 2k$ (this is true for $k = 3$), but Lewin showed that it is false for large $k$. Posa proved (unpublished) that every $g(n;[kn])$ contains a circuit with at least $ck^2$ diagonals.

Let $f_r(n)$ be the smallest integer for which every $G(n;f_r(n))$ contains a $K_r$. Is there a constant $c_r$ so that every $G(n;f_r(n))$ has a vertex $x_1$ of valency $m > c_r n$ so that the graph spanned by its star has at least $f_{r-1}(m)$ edges? (The star of the vertex is the set of vertices joined to it.) If true this would be a nice generalization of Turán's theorem. The first interesting case is $r = 4$ and I could not settle this case.

Let $G$ be a bipartite graph of $n$ vertices with $[n^{2/3}]$ black and $n - [n^{2/3}]$ white vertices. Is it true that if the number of edges is greater than $cn$ then our graph contains a $C_6$? It is easy to see that it contains a $C_8$.

Finally I state an old conjecture of Hajnal and myself: Consider a $G(n;[kn])$ and let $3 \le r_1 < r_2 < \dots < r_n \le n$ be the set of integers $r$ for which our $G(n;[kn])$ contains a $C_r$. Determine or estimate

$$\min \sum \frac{1}{r_i}$$

where the minimum is extended over all $G(n;[kn])$. It seems likely that the minimum is $(\frac{1}{2} + o(1)) \log k$ but we could not even prove that it tends to infinity as $k$ tends to infinity (independently of $n$).