CONSECUTIVE INTEGERS

by P. Erdos

Very recently two old problems on consecutive integers were settled. Catalan conjectured that 8 and 9 are the only consecutive powers. First of all observe that four consecutive integers cannot all be powers since one of them is congruent to 2 modulo 4. It is considerably more difficult to prove that three consecutive integers can not all be powers; this was accomplished about twenty years ago by Cassels and Makowski. Finally in 1974 using some deep results of Baker, Tijdeman proved that there is an n_0, whose value can be given explicitly, such that for $n > n_0$ n and $n+1$ are not both powers. This settles Catalan's conjecture nearly completely, and there is little doubt that it will be settled in full soon. It has been conjectured that if $x_1 < x_2 < x_3 \ldots$ is a sequence of consecutive powers, $x_1 = 1, x_2 = 4, \ldots$ then $x_{i+1} - x_i > i^c$ for all i and some absolute constant c. At the moment this seems intractable. (The paper of Tijdeman will appear in Acta Arithmetica.)

It was conjectured more than a century ago that the product of consecutive integers is never a power. Almost 40 years ago, Rigge and I proved that the product of consecutive integers is never a square, and recently Selfridge and I proved the general conjecture. In fact, our result is, that for every k and l there is a prime $p \geq k$ so that if

$$p^{\alpha_{k,l}} \,||\, \Pi_1^k (n+i)$$

then

$$\alpha_{k,l} \not\equiv \text{mod.}(l).$$

We conjecture that in fact for all $k > 2$ there is a prime $p \geq k$ with $a_{k,l} = 1$, but this is also intractable at the moment.

It often happens in number theory that every new result suggests many new questions - which is a good thing as it ensures that the supply of Mathematics is inexhaustible! I would now turn to discuss a few more problems and results on consecutive integers and in particular a simple conjecture of mine which is more than 25 years old.

Put

$$m = a_k(m) b_k(m),$$

$$a_k(m) = \Pi p^{\alpha_p}$$

where the product extends over all the primes $p \geq k$ and $p^\alpha \,||\, m$. Further define

$$f(n;k,l) = \min\{a_k(n+i) \,|\, 1 \leq i \leq l\}$$

$$F(k,l) = \max\{f(n;k,l) \,|\, 1 \leq n < \infty\}.$$

I conjectured that

1) $$\lim_{k\to\infty} F(k,k)/k = 0$$

In other words, is it true that for every ϵ there is a k_ϵ such that for every $k \geq k_\epsilon$ at least one of the integers $a_1(n+i)$, $i=1,\ldots,l$, is less than k_ϵ. I am unable to prove this but will outline the proof of

2) $$F(k,k) < (1+\epsilon)k \text{ for } k > k_0(\epsilon).$$

To prove (2) consider

3) $$A(n,k) = \tilde{\Pi}_1^k a_1(n+i)$$

where in (3) the tilde indicates that for every $p \leq k$ we omit one of the integers $n+i$ divisible by a maximal power of p. Then the product $\tilde{\Pi} a_k(n+i)$ has at least $k-\pi(k)$ factors and by a simple appilication of the Legendre formula for the factorisation of $k!$ we obtain

4) $$\tilde{\Pi} a_k(n+i) \,|\, k!\,.$$

If (2) did not hold, we have from (4) and Stirling's formaul

5) $$((1+\varepsilon)k)^{k-\pi(k)} < k^{k+1}\exp(-k)$$

or $$k^{\pi(k)+1} > \exp(k)(1+\varepsilon)^{k-\pi(k)}$$

Now, by the prime number theorem,

$$\pi(k) < \frac{(1+\varepsilon/10)k}{\log k}$$

and so from (5),

$$k^{\left((1+\varepsilon/10)\frac{k}{\log k}+1\right)} > \exp(k).(1+\varepsilon)^{\left(k-\frac{2k}{\log k}\right)}$$

which is false if k is large enough, and this contradiction proves (2).

Assume for the moment that (1) has been proved. Then one can immediately ask for the true order of magnitude of F(k,k). I expect that it is $o(k^{\varepsilon})$ for every $\varepsilon>0$. On the other hand, I can prove that

6) $$F(k,k) > \exp\left(c.\frac{\log(k)\log\log\log(k)}{\log\log(k)}\right)$$

The problem of estimating F(k,k) and the proof of (6) is connected with the following question on the seive of Eratosthenes-Prim-Selberg : determine or estimate the sm smallest integer A(k) so that one can find, for every p with $A(k) \leq p \leq k$, a residue u_p such that for every integer $t \leq k$, t satisfies one of the congruences to u_p modulo p. Clearly $F(k,k) \nmid A(k)$. Using the method of Rankin-Chen and myself I proved

7) $$A(k) > \exp(c.\log(k)\log\log\log(k)/\log(k))$$

which implies 6. I do not give the proofs here. It would be interesting and useful to prove $A(k) < k^{\varepsilon}$ for every $\varepsilon>0$ and sufficiently large k.

Now, I shall say a few words about F(k,l) for $k \neq l$. It follows easily from the Chinese Remainder Theorem that for $l \leq \pi(k)$ we have $F(k,l) = \infty$, since for a suitable n, we can make $n+i$, $1 \leq i \leq \pi(k)$ divisible by an arbitrarily large power of p_1. It is easy to see that this no longer holds for $l = \pi(k)+1$ and in fact it is not hard to prove that

$$F(k,\pi(k)+1) = \Pi p^{\alpha_p}$$

where $$p^{\alpha_p} \leq \pi(k) < p^{\alpha_p+1}.$$

As l increases it gets much harder to even estimate $F(k,l)$. Many more problems can be formaulated which I leave to the reader and only state one which is quite fundamental:

Determine or estimate the least $l = l_k$ so that $F(k,l_k)=1$.

In other words, the least l_k so that among l_k consecutive integers there is always one relatively prime to the primes less than k. This question is of course connected with the problem of estimating the differnce of consecutive primes and also with the following problem of Jacobsthal: Denote by $g(m)$ the least integer so that any set of $g(m)$ consecutive integers contains one which is relatively prime to m. At the recent meeting on Number Theory in Oberwolfach (Nov.'75) Kanold gave an interesting talk on $g(m)$ and the paper will appear soon. Vaughan observed that the seive of Rosser gives $g(m) < (\log(m))^{(2+\epsilon)}$ for all $\epsilon > 0$ if m is sufficiently large. The true order of magnitude is not known.

It seems to me that interesting and difficult problems remain for $l \leq \pi(k)$ too. Here we have to consider the dependence on n too. It is not hard to show that for every $\epsilon>0$ there are infinitley many values of n for which

$$f(n;k,l) > (1-\epsilon)^{l}\sqrt{n}\,. \tag{8}$$

The proof of (8) uses some elemtry facts of Diophantine approximation and the Chinese Remainder Theorem. We do not give the details. I do not know how much (8) can be improved. By a deep theorem of Mahler, using the p-adic Thue-Siegel Theorem, $f(n;k,l) > n^{(\varepsilon+1/l)}$. It is quite possible that

9) $$\limsup_{n\to\infty} f(n;k,l)^{l}/n = \infty.$$

Interesting problems can also be raised if k tends to infinity with n; e.g. how large can $f(n;k,\pi(k))$ become if $k = (1+o(1))\log(n)$? It seems to be difficult to write a really short note on the subject since new problems occur while one is writing!

It would be of some interest to know how many of the integers $a_k(n+i)$ must be different. I expect that more than c.k are. If this is proved one of course must determine the best possible value of c.

Denote by K(l) the greatest integer below l composed entirely of primes below k. Trivially

10) $$\min_{n} \max_{i} a_k(n+i) = K(l)$$

To prove (10) observe that on the one hand any set of l consecutive integers contains a multiple of K(l), on the other that if 2l divides t, then the integers t!+1,...,t!+l clearly satisfy (10), when n=0. More generally, try to characterise the set of n which satisy (10). To simplify matters, let k=l and denote n_k as the smallest positive integer with $\max_i a_k(n+i) = k$, S_k as the class of all integers n such that this is true. If p^{a_p} is the greatest power of p not exceeding k then

$$\prod_{p\leq k} p^{a_p+1} \in S_k .$$

Perhaps I am overlooking an obvious explicit construction for n_k but at the moment I do not even have good upper or lower bounds for it. When is k! in S_k , The smallest such k is 8 and I do not know if there are infinitely many such k's. By Wilson's theorem, p! is never in S_p.

To complete this note, I state three more extremal problems in number theory. Put

$$n! = \Pi a_i,\ a_1 \le a_2 \le a_3 \ldots \le a_n.$$

Determine $\max\{a_1\}$. It follows easily from Stirling's formula that a_1 does not exceed $(n/e)(1-c/\log(n))$. I conjectured that for every $\eta>0$ and suffciently large n, max a_1 exceeds $(1-\eta)n/e$.

Put

$$n! = \Pi b_i,\ 1<b_1<b_2<\ldots<b_k \le n$$

Determine or estimate min k.

Clearly k exceeds $n - n/\log(n)$ and by more complicated methods I can prove

$$k = n - (1+o(1))n/\log(n)$$

$$k > n - n(\log(n)+c)/(\log(n))^2$$

where c is a positive absolute constant.

Put

11) $$n! = \Pi u_i,\ u_1<u_2<\ldots<u_k$$

Determine or estimate min u_k - k is not fixed. It is not hard to prove that u_k less than 2n has only a finite number of solutions. I only know of two:

$$6! = 8.9.10$$

and $$14! = 16.21.22.24.25.26.27.28\ .$$

It would be difficult to deterimne all the solutions, although Vaughan has just found some more -

$$3! = 6$$

$$8! = 12.14.15.16$$

$$11! = 15.16.18.20.21.22$$

$$15! = 16.18.20.21.22.25.26.27.28$$

and this is all up to 15. Vaughan also tells me

$$40! = 42.44.45.48.49.50.51.52.54.55.56.57.$$

$$.58.59.60.62.63.64.65.66.68.69.72.74.80$$