JOURNAL OF NUMBER THEORY 6, 52–63 (1974)

On the Distribution of Values of Certain Divisor Functions

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Communicated July 7, 1971

Let $\{\epsilon_d\}$ be a sequence of nonnegative numbers and $f(n) = \sum \epsilon_d$, the sum being over divisors d of n. We say that f has the distribution function F if for all $c > 0$, the number of integers $n \leqslant x$ for which $f(n) > c$ is asymptotic to $xF(c)$, and we investigate when F exists and when it is continuous.

Let $\{\epsilon_d\}$ be a sequence of nonnegative numbers and

$$f(n) = \sum_{d \mid n} \epsilon_d .$$

Is it true that for all $c \geqslant 0$,

$$\sum_{\substack{n \leqslant x \\ f(n) > c}} 1 \sim xF(c)$$

for some function $F(c)$ depending only on the value of c? If so, it is plain that $0 \leqslant F(c) \leqslant 1$; moreover, F is nonincreasing. If ϵ_d is large enough, say $\epsilon_d = 1$ for all d so that $f(n) = \tau(n)$, then $F(c) = 1$ identically. Therefore, it is interesting to ask under what circumstances F exists and

$$\operatorname*{Lt}_{c \to \infty} F(c) = 0.$$

In this case, we say that f has the *distribution function F*. We prove the following:

THEOREM. *The result holds if*[1]

$$\epsilon_d = 1/(\log d)^\alpha \qquad \text{or} \qquad \epsilon_d = 2^{-\log\log d-(1+\beta)(2\log\log d\cdot\log\log\log\log d)^{1/2}}$$

for every $\alpha > \log 2$ *and* $\beta > 0$. *F is continuous and tends to zero as c tends to infinity; in fact, as* $\delta \to 0$, *we have that*

$$F(c-\delta) - F(c) \ll (\log(1/\delta))^{-1/2}.$$

Here the constant implied by Vinogradov's notation $\ll$ *is independent of c. The lower bound* $\log 2$ *is best possible: if* $\alpha = \log 2$, *then the normal order of* $f(n)$ *tends to infinity with n. The second form of* ϵ_d *shows precisely how large it can be; in this case, the normal order of* $f(n)$ *tends to infinity if* $\beta < 0$.

We also show that in the case

$$f(n; q, a) = \sum_{\substack{d \mid n \\ d \equiv a \pmod q}} \epsilon_d, \qquad (a, q) = 1,$$

we have

$$\sum_{\substack{n \leq x \\ f(n; q, a) > c}} 1 \sim xF(c; q, a),$$

where $F(c; q, a)$ has similar properties to $F(c)$. It would be interesting to know how $F(c; q, a)$ varies with q and a, and we hope to investigate this question in a later paper. We now give the

Proof of the Theorem. We let

$$f_k(n) = \sum_{d \mid n} \epsilon_d, \qquad d \text{ has no prime factor} > k.$$

Since

$$\sum_{\substack{n=1 \\ f_k(n) > c}}^{\infty} \frac{1}{n^s} = \zeta(s) \prod_{p \leq k} \left(1 - \frac{1}{p^s}\right) \sum_{m \in M_k(c)} \frac{1}{m^s},$$

where $M_k(c)$ is the set of integers m having no prime factor $> k$ and for which $f(m) = f_k(m) > c$, we have

$$\sum_{\substack{n \leq x \\ f_k(n) > c}} 1 \sim xF_k(c)$$

[1] To ensure that the iterated logarithm is well-defined for small values of the variable, moreover that ϵ_1 is finite, it is understood throughout that $\log x$ is to be interpreted as $\max(\log x, 1)$.

for all $c \geqslant 0$, and

$$F_k(c) = \prod_{p \leqslant k} \left(1 - \frac{1}{p}\right) \sum_{m \in M_k(c)} \frac{1}{m}.$$

The sequence $\{F_k(c)\}$ is monotonic increasing and bounded above by 1. Hence,

$$0 \leqslant F^*(c) = \operatorname*{Lt}_{k \to \infty} F_k(c) \leqslant 1$$

is well-defined and is the intuitive value of $F(c)$ if F exists. We start by looking for upper and lower bounds for the sum

$$\sum_{\substack{n \leqslant x \\ f(n) > c}} 1.$$

As it is rather easier, we begin with the

Lower Bound. Since $f(n) \geqslant f_k(n)$, we have for all k that

$$\sum_{\substack{n \leqslant x \\ f(n) > c}} 1 \geqslant \sum_{\substack{n \leqslant x \\ f_k(n) > c}} 1$$

$$\geqslant \sum_{n \leqslant x} \sum_{\substack{m \mid n \\ m \in M_k(c) \\ (n/m, P(k)) = 1}} 1,$$

where $P(k)$ is the product of all primes $\leqslant k$. This is

$$\sum_{m \in M_k(c)} \sum_{\substack{r \leqslant x/m \\ (r, P(k)) = 1}} 1 \geqslant \sum_{\substack{m \in M_k(c) \\ m \leqslant H}} \left(\frac{x}{m} \prod_{p \leqslant k} \left(1 - \frac{1}{p}\right) - 2^{\pi(k)}\right)$$

for any value of H. We choose this rather less than x to limit the error term arising from the $2^{\pi(k)}$. This is

$$\geqslant xF_k(c) - 2^{\pi(k)}H - x \prod_{p \leqslant k} \left(1 - \frac{1}{p}\right) \sum_{\substack{m > H \\ m \in M_k(c)}} \frac{1}{m}.$$

The last sum on the right does not exceed

$$\frac{1}{H^{1/2}} \prod_{p \leqslant k} \left(1 - \frac{1}{p^{1/2}}\right)^{-1} \leqslant \frac{1}{H^{1/2}} \exp\left(\frac{A_1 k^{1/2}}{\log k}\right),$$

where A_1 is an absolute constant. We select $H = x^{2/3}$, and we deduce that

$$\sum_{\substack{n \leqslant x \\ f(n) > c}} 1 \geqslant xF_k(c) + O(x^{2/3} 2^{\pi(k)}).$$

If now $k \to \infty$ with x so that $2^{\pi(k)} = o(x^{1/3})$, we have

$$\sum_{\substack{n \leqslant x \\ f(n) > c}} 1 \geqslant x(F^*(c) + o(1)) + o(x) = xF^*(c) + o(x).$$

As a particular case, if $F^*(c) = 1$ identically, then F exists and $F(c) = 1$ for all c. Note that so far we have only used the fact that $\epsilon_d \geqslant 0$ for all d.

Upper Bound. For all $k > 0$ and $\delta > 0$, we have

$$\sum_{\substack{n \leqslant x \\ f(n) > c}} 1 \leqslant \sum_{\substack{n \leqslant x \\ f_k(n) > c - \delta}} 1 + \sum_{\substack{n \leqslant x \\ f(n) - f_k(n) \geqslant \delta}} 1.$$

Examining the first sum on the right, we have

$$\sum_{\substack{n \leqslant x \\ f_k(n) > c - \delta}} 1 = \sum_{\substack{m \leqslant x \\ m \in M_k(c-\delta)}} \sum_{\substack{r \leqslant x/m \\ (r, P(k)) = 1}} 1$$

$$\leqslant \sum_{\substack{m \leqslant H \\ m \in M_k(c-\delta)}} \left(\frac{x}{m} \prod_{p \leqslant k} \left(1 - \frac{1}{p}\right) + 2^{\pi(k)} \right) + x \sum_{m > H}' \frac{1}{m},$$

the last sum being restricted to m's having no prime factor exceeding k. This is

$$\leqslant xF_k(c - \delta) + 2^{\pi(k)}H + \frac{x}{H^{1/2}} \exp\left(\frac{A_1 k^{1/2}}{\log k}\right)$$

and, as before, we select $H = x^{2/3}$ and require that

$$2^{\pi(k)} = o(x^{1/3}).$$

For this range of values of k, we deduce that

$$\sum_{\substack{n \leqslant x \\ f(n) > c}} 1 \leqslant xF^*(c) + x\{F_k(c - \delta) - F_k(c)\} + \sum_{\substack{n \leqslant x \\ f(n) - f_k(n) \geqslant \delta}} 1 + o(x).$$

We have to show that if $k \to \infty$ and $\delta \to 0$ as $x \to \infty$, then $F_k(c - \delta) - F_k(c) = o(1)$, and our method also shows that F is continuous. Now

$$F_k(c - \delta) - F_k(c) = \prod_{p \leqslant k} \left(1 - \frac{1}{p}\right) \sum_{c - \delta < f(m) \leqslant c}' \frac{1}{m},$$

all the prime factors of m being $\leqslant k$. Since

$$f(md) \geqslant f(m) + \sum_{p \mid d,\, p \nmid m} \epsilon_p,$$

if d has any prime factor not dividing m for which $\epsilon_p \geqslant \delta$, not both m and md contribute to $\sum'$. Let

$$Q(k, \delta) = \{p; p \leqslant k \text{ and } \epsilon_p \geqslant \delta\}$$

and $R(k, \delta)$ be the maximal sum of the form

$$\sum'' 1/d$$

where every prime factor of d belongs to $Q(k, \delta)$ and if d_1 and d_2 both contribute to $\sum''$ and $d_1 \mid d_2$, then d_2 has no prime factor not dividing d_1. Then

$$\sum_{c-\delta<f(m)\leqslant c}' 1/m \leqslant \prod_{\substack{p\leqslant k \\ p\notin Q(k,\delta)}} \left(1-\frac{1}{p}\right)^{-1} R(k, \delta)$$

and

$$F_k(c-\delta) - F_k(c) \leqslant \prod_{p\in Q(k,\delta)} \left(1-\frac{1}{p}\right) R(k, \delta).$$

Now let $\tau_k''(n)$ denote the number of divisors d of n which contribute to the maximal sum $R(k, \delta)$. Then for $y \geqslant 0$,

$$\begin{aligned} yR(k, \delta) &\geqslant \sum_{n\leqslant y} \tau_k''(n) = \sum_{d\leqslant y}'' \left[\frac{y}{d}\right] \\ &\geqslant y\left\{R(k, \delta) - \sum_{d>y}'' \frac{1}{d}\right\} - \sum_{d\leqslant y}'' 1 \\ &\geqslant yR(k, \delta) - 2y^{1/2} \prod_{p\in Q(k,\delta)} \left(1-\frac{1}{p^{1/2}}\right)^{-1}, \end{aligned}$$

and therefore

$$R(k, \delta) = \operatorname*{Lt}_{y\to\infty} \frac{1}{y} \sum_{n\leqslant y} \tau_k''(n).$$

Now let $n = mh$, where m is the largest divisor of n all of whose prime factors belong to $Q(k, \delta)$. Thus

$$\tau_k''(n) = \tau_k''(m).$$

By a result of de Bruijn, Tengbergen, and Kruyswijk [2], we may split the divisors of m into disjoint symmetric chains. A chain is a sequence of integers each dividing the next, the quotient being a prime; it is symmetric in the sense that the total number of prime factors of its first and last members equals the number of prime factors of m. Ian Anderson [3] showed that the number of chains is

$$\ll \tau(m)/\omega(m)^{1/2}.$$

Now suppose that two divisors d_1, d_2 of n (and so of m) contributing to $R(k, \delta)$ belong to the same chain, so that one divides the other, say $d_1 \mid d_2$. Then d_1 and d_2 have the same prime factors. Hence, $\tau_k''(m)$ does not exceed the number of chains times the maximal number of divisors of m all of which have the same prime factors. If

$$m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}, \qquad \hat{m} = p_1 p_2 \cdots p_r,$$

this is $\alpha_1 \alpha_2 \cdots \alpha_r = \tau(m/\hat{m})$. Therefore

$$\tau_k''(n) \ll \frac{\tau(m)\, \tau(m/\hat{m})}{(\omega(m))^{1/2}}.$$

Hence for any $H > 0$,

$$\sum_{n \geqslant y} \tau_k''(n) \ll 2^H y + \frac{1}{H^{1/2}} \sum_{n \leqslant y} \tau(m)\, \tau(m/\hat{m}).$$

Now

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\tau(m)\, \tau(m/\hat{m})}{n^s} &= \prod_{p \notin Q(k,\delta)} \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{p \in Q(k,\delta)} \left(1 + \frac{1 \cdot 2}{p^s} + \frac{2 \cdot 3}{p^{2s}} + \cdots\right) \\
&= \zeta(s) \prod_{p \in Q(k,\delta)} \left(1 + \frac{1}{p^s} + \frac{4}{p^{2s}} + \frac{6}{p^{3s}} + \frac{8}{p^{4s}} + \cdots\right) \\
&= \zeta(s) \prod_{p \in Q(k,\delta)} \left(1 - \frac{1}{p^s}\right)^{-1} \cdot \left(1 + \frac{3}{p^{2s}} + \frac{2}{p^{3s}} + \frac{2}{p^{4s}} + \cdots\right)
\end{aligned}$$

so that

$$\sum_{n \leqslant y} \tau(m)\, \tau(m/\hat{m}) \sim y \prod_{p \in Q(k,\delta)} \left(1 - \frac{1}{p}\right)^{-1} \left(1 + \frac{3p - 1}{p^3 - p^2}\right).$$

Setting

$$H = \sum_{p \in Q(k,\delta)} \frac{1}{p},$$

we deduce that

$$\sum_{n \leqslant y} \tau_k''(n) \ll y \left(\sum_{p \in Q(k,\delta)} \frac{1}{p}\right)^{-1/2} \prod_{p \in Q(k,\delta)} \left(1 - \frac{1}{p}\right)^{-1}$$

so that

$$F_k(c-\delta) - F_k(c) \leqslant T(k, \delta) = \Big(\sum_{p \in Q(k,\delta)} \frac{1}{p}\Big)^{-1/2}.$$

We now have that

$$\sum_{\substack{n \leqslant x \\ f(n) > c}} 1 \leqslant xF^*(c) + \sum_{\substack{n \leqslant x \\ f(n) - f_k(n) \geqslant \delta}} 1 + O(xT(k, \delta)) + o(x).$$

To ensure that $T(k, \delta) \to 0$ as $\delta \to 0$ and $k \to \infty$ with x, we require only

Condition 1. The series

$$\sum_{\epsilon_p > 0} \frac{1}{p}$$

is divergent.

This is of course satisfied by the sequence $\{\epsilon_d\}$ in the theorem. We also introduce

Condition 2. If p is a prime, then $\epsilon_{pm} \leqslant \epsilon_p$ for all integers m.

This is convenient and requires rather less than that the sequence $\{\epsilon_d\}$ is nonincreasing, although both those under consideration are.

Now let d be a divisor of n whose prime factors all exceed k, and t a divisor none of whose prime factors exceed k. Clearly, every divisor of n can be written uniquely in the form dt, and so

$$f(n) - f_k(n) = \sum_{\substack{d \mid n \\ d \neq 1}} \sum_{t \mid n} \epsilon_{dt}.$$

Next, assume that n has no repeated prime factor exceeding k. The number of exceptional $n \leqslant x$ is

$$\leqslant \sum_{p > k} \frac{x}{p^2} = O\left(\frac{x}{k \log k}\right) = o(x)$$

if $k \to \infty$ with x. If

$$\tau_k(n) = \sum_{t \mid n} 1,$$

then by Condition 2, we have

$$f(n) - f_k(n) \leqslant \tau_k(n)\{\epsilon_{p_1} + 2\epsilon_{p_2} + 4\epsilon_{p_3} + \cdots + 2^{m-1}\epsilon_{p_m}\},$$

where $p_1, p_2, \ldots, p_m$ are the prime factors of n exceeding k in any order; naturally, it is advantageous to select the order for which

$$\epsilon_{p_1} \geqslant \epsilon_{p_2} \geqslant \epsilon_{p_3}, \ldots, \geqslant \epsilon_{p_m},$$

so that in the present application, $p_1, p_2, \ldots, p_m$ are simply in increasing order.

We need the following lemma, which is an application of Theorem VI of Erdös [1].

LEMMA. *Let $\nu_y(n)$ denote the number of distinct prime factors of n not exceeding y and λ be fixed > 0. Then provided $y_0 \geqslant y_0(\lambda)$, the numbers n for which*

$$|\nu_y(n) - \log\log y| \leqslant (1 + \lambda)(2 \log\log y \cdot \log\log\log\log y)^{1/2}$$

for all y, $y_0 \leqslant y \leqslant n$, have a positive density; moreover, as $y_0 \to \infty$, this density tends to 1.

We apply this as follows: We let $y_0 = k$ which tends to infinity with x; therefore, the lemma applies to almost all $n \leqslant x$. We take $p_1, p_2, \ldots, p_m$ to be in increasing order. Then for almost all $n \leqslant x$ and each i, $i \leqslant m$, we have

$$i + \nu_k(n) = \nu_{p_i}(n) \leqslant \log\log p_i + (1 + \lambda)(2 \log_2 p_i \cdot \log_4 p_i)^{1/2}$$

using the notation $\log_{i+1} x = \log(\log_i x)$ for iterated logarithms. We choose λ strictly less than the β given in the theorem; say $\lambda = \beta/2$.

We will prove the theorem only for the second form of ϵ_d as the other is treated similarly, except that we may use a weaker version of the above lemma which can be obtained from the familiar variance argument due to Turán. In the present case, since

$$\epsilon_p = 2^{-\log\log p-(1+\beta)(2\log_2 p\cdot\log_4 p)^{1/2}},$$

we have

$$\sum_{i=1}^{m} 2^{i-1}\epsilon_{p_i} = \sum_{i=1}^{m} 2^{i-1-\log\log p_i-(1+2\lambda)(2\log_2 p_i\cdot\log_4 p_i)^{1/2}}$$

$$\leqslant 2^{-\nu_k(n)} \sum_{i=1}^{m} 2^{-\lambda(2\log_2 p_i\cdot\log_4 p_i)^{1/2}}.$$

We may assume that for each i,

$$2 \log\log p_i \geqslant i + \nu_k(n),$$

and since

$$\sum_{i=1}^{\infty} 2^{-\xi(i+\nu)^{1/2}} \leqslant \int_{\nu}^{\infty} 2^{-\xi t^{1/2}}\,dt \leqslant 2^{-(1/2)\xi\nu^{1/2}} \int_{0}^{\infty} 2^{-(1/2)\xi t^{1/2}}\,dt \ll \frac{2^{-(1/2)\xi\nu^{1/2}}}{\xi^2},$$

setting

$$\nu = \nu_k(n), \qquad \xi = \lambda(\log_4 p_1)^{1/2} \geqslant \tfrac{1}{2}\lambda(\log\log \nu_k(n))^{1/2},$$

we obtain

$$\sum_{i=1}^{m} 2^{i-1}\epsilon_{\nu_i} \leqslant \frac{A_4}{\lambda^2}\, 2^{-\nu_k(n)-(1/4)\lambda(\nu_k(n)\log\log\nu_k(n))^{1/2}}.$$

It follows that if $\omega_k(n)$ denotes the number of prime factors of n not exceeding k and counted according to multiplicity, then for almost all $n \leqslant x$,

$$f(n) - f_k(n) \leqslant (A_4/\lambda_2)\, 2^{\omega_k(n)-\nu_k(n)-(1/4)\lambda(\nu_k(n)\log\log\nu_k(n))^{1/2}}.$$

Since $k \to \infty$ with x, for almost all $n \leqslant x$, we have that

$$\nu_k(n) \geqslant (1/2)\log\log k.$$

Also,

$$\omega_k(n) - \nu_k(n) \leqslant (\lambda/20)(\log_2 k \cdot \log_4 k)^{1/2} \leqslant (\lambda/8)(\nu_k(n)\log\log \nu_k(n))^{1/2}.$$

To see this, note that

$$\sum_{n\leqslant x} \{\omega_k(n) - \nu_k(n)\} = \sum_{p\leqslant k} \left[\frac{x}{p^2}\right] + \left[\frac{x}{p^3}\right] + \cdots \leqslant x\sum_{p} \frac{1}{p(p-1)} \leqslant x.$$

Therefore, the number of integers $n \leqslant x$ for which $\omega_k(n) - \nu_k(n) \geqslant h$ does not exceed x/h. If

$$h = (\lambda/20)(\log_2 k \cdot \log_4 k)^{1/2},$$

this is $o(x)$ as $k \to \infty$ with x. Therefore, for almost all $n \leqslant x$,

$$f(n) - f_k(n) \leqslant (A_4/\lambda^2)\, 2^{-(\lambda/20)(\log_2 k\cdot\log_4 k)^{1/2}}$$

If $\delta \to 0$ more slowly than this, we deduce that

$$\sum_{\substack{n\leqslant x \\ f(n)-f_k(n)\leqslant\delta}} = o(x).$$

We deduce that

$$\sum_{\substack{n \leqslant x \\ f(n) > c}} 1 \leqslant xF^*(c) + o(x),$$

and combining this with the lower bound result, we get

$$\sum_{\substack{n \leqslant x \\ f(n) > c}} 1 \sim xF(c), \qquad F = F^*.$$

Next, we show that F is continuous. We know that

$$F_k(c - \delta) - F_k(c) \ll T(k, \delta),$$

the constant implied by Vinogradov's notation $\ll$ being uniform in k, c, and δ. Letting $k \to \infty$, $Q(k, \delta)$ becomes

$$\{p;\ \epsilon_p \geqslant \delta\}.$$

Hence

$$\operatorname*{Lt}_{k \to \infty} T(k, \delta) \ll (\log(1/\delta))^{-1/2}$$

for either form of $\{\epsilon_q\}$. Therefore F is continuous, indeed uniformly. It remains to show that

$$\operatorname*{Lt}_{c \to \infty} F(c) = 0.$$

We do this by a treatment of $f(n) - f_k(n)$ similar to the above, but replacing "almost all $n \leqslant x$" by "for all but at most ϵx integers $n \leqslant x$" at each step. Given any $\epsilon > 0$, there exists a k so large that on a sequence of integers of density at least $1 - \epsilon$, we have

$$f(n) - f_k(n) \leqslant (A_4/\lambda^2)\, 2^{-(\lambda/20)(\log_2 k - \log_4 k)^{1/8}} \leqslant (A_4/\lambda_2).$$

Also

$$\sum_{n \leqslant x} \tau_k(n) \leqslant x \prod_{p \leqslant k} \left(1 - \frac{1}{p}\right)^{-1} \leqslant A_5 x \log k.$$

Hence, the integers for which

$$\tau_k(n) \geqslant (A_5/\epsilon) \log k$$

have density not exceeding ϵ. Therefore, on a sequence of density $\geqslant 1 - 2\epsilon$, we have

$$f(n) \leqslant \tau_k(n) + (f(n) - f_k(n)) \leqslant (A_4/\lambda^2) + (A_5/\epsilon) \log k.$$

Setting

$$c = c(\epsilon) = (A_4/\lambda^2) + (A_5/\epsilon)\log k, \qquad k = k(\epsilon),$$

we deduce that

$$F(c) \leqslant 2\epsilon,$$

giving the result stated.

We conclude by deducing a similar result for $f(n; q, a)$. We set

$$\epsilon_d' = \begin{cases} \epsilon_d & \text{if} \quad d \equiv a \pmod{q}, \\ 0 & \text{otherwise}. \end{cases}$$

The treatment of the lower bound goes through as before, and that of the upper bound is largely unaltered, for we have

$$f(n; q, a) - f_k(n; q, a) \leqslant f(n) - f_k(n),$$

and so it is clear that

$$\sum_{\substack{n \leqslant x \\ f(n;q,a) - f_k(n;q,a) \geqslant \delta}} 1 \leqslant \sum_{\substack{n \leqslant x \\ f(n) - f_k(n) \geqslant \delta}} 1 = o(x)$$

from the above. In the treatment of $F_k(c - \delta; q, a) - F_k(c; q, a)$, we have to consider

$$Q(k, \delta, q, a) = \{p \colon p \leqslant k \text{ and } \epsilon_p \geqslant \delta, p \equiv a \pmod{q}\}.$$

The argument goes through as before: we require that the series

$$\sum_{\epsilon_p' > 0} \frac{1}{p} = \sum_{p \equiv a \pmod{q}} \frac{1}{p}$$

diverges, and since $(a, q) = 1$, this is the case.

A similar argument gives the following more general result: If

$$0 \leqslant \epsilon_d \leqslant 2^{-\log\log d - (1+\beta)(2\log\log d \cdot \log\log\log\log d)^{1/2}}, \qquad \beta > 0,$$

and Condition 1 holds, then f has a continuous distribution function.

It seems possible that Condition 1 may be weakened; also, we should like to consider the case where ϵ_d may be negative. We leave these questions to a later paper.

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