PACIFIC JOURNAL OF MATHEMATICS
Vol. 52, No. 1, 1974

ON THE DISTRIBUTION OF NUMBERS OF THE FORM $\sigma(n)/n$ AND ON SOME RELATED QUESTIONS

Dedicated to my friend I. Schoenberg on the occasion of his 70th birthday

P. ERDÖS

A number theoretic function $f(n)$ is called multiplicative if $f(ab) = f(a)f(b)$ for $(a, b) = 1$, it is called additive if $f(a\,b) = f(a) + f(b)$ for $(a, b) = 1$. A function $f(n)$ is said to have a distribution function if for every c the density $g(c)$ of integers satisfying $f(n) < c$ exists and $g(-\infty) = 0, g(\infty) = 1$.

In this note we give some best possible estimates for $g(c + 1/t) - g(t)$, for the case of $f(n) = \sigma(n)/n$.

More than 40 years ago I. Schoenberg proved that $\phi(n)/n$ ($\phi(n)$ is Euler's ϕ function) has a continuous distribution function [12]. This result was the starting point of a systematic theory of additive and multiplicative functions. Very soon Behrend, Chowla, and Davenport [2] proved that $\sigma(n)/n$ ($\sigma(n) = \sum_{d\mid n} d$) also has a continuous distribution function. Thus it followed that the density of abundant numbers $g(2)$ exists. (An integer n if abundant if $\sigma(n)/n \geqq 2$, otherwise it is deficient.) The value $g(2)$ of this density is known only with very poor accuracy, it seems to be fairly close to $1/4$ but is not equal to it [1].

I do not discuss here general theory of the distribution of values of additive and multiplicative functions, just remark that necessary and sufficient conditions are known for the existence and continuity of the distribution function of additive and multiplicative functions [4], but relatively little is known about absolute continuity. In 1939, Aurel Wintner called my attention to the problem of absolute continuity of the distribution function of additive and multiplicative functions. I proved (among others) that the distribution function of $\sigma(n)/n$ and $\phi(n)/n$ is purely singular, but that there are additive (and multiplicative) functions whose distribution function is an entire function [5]. No necessary and sufficient condition for the absolute continuity of the distribution function seems to be known and e.g., it is not known if the distribution function of the additive function $f(p) = 1/\log p$ is absolutely continuous.

Denote by $g(c)$ the distribution function of $\sigma(n)/n$. Since $g(c)$ is a purely singular monotonic function its derivative is almost everywhere 0. As far as I know it is not known if the derivative can take any other value. It is easy to see that the derivative from the right of $g(c)$ for $c = \sigma(n)/n$ is infinity, but it is doubtful if the derivative from the left exists. I do not know if the derivative from the right (or left) can take any value other than 0 or infinity. It is easy to see that there is a dense set of values of c for which the derivative does not exist from the left and from the right.

Two numbers a and b are called amicable if $\sigma(a) = \sigma(b) = a + b$. I proved [6] that the density of integers which occur in an amicable pair is 0. On the other hand, it is not yet known if the number of amicable pairs is infinite. Rieger obtained an explicit upper bound for the number of integers not exceeding x which occur in an amicable pair and in this connection asked me to obtain as sharp an estimation as possible for $F(x; a, b)$ the number of integers $n \leqq x$ satisfying

$$a \leqq \frac{\sigma(n)}{n} < b .$$

I prove the following

THEOREM. *There is an absolute constant* c_1 *so that for* $0, x > t$

$$F\left(x; a, a + \frac{1}{t}\right) < c_1 \frac{x}{\log t} . \tag{1}$$

Apart from the value of c_1*, this inequality is best possible.*

This sharpens a result of Tyan [13]. The same results hold also if $\sigma(n)$ is replaced by Euler's ϕ function, in fact the proofs are a little simpler. Incidentally with a little trouble we could prove instead of (1) the following slightly stronger

$$F\left(x; a, a\left(1 + \frac{1}{t}\right)\right) < c_1 x/\log t . \tag{1'}$$

Using (1) and (1′) we can deduce (following Diamond [3]) that

$$F(x; 1, a) = xg(a) + o\left(\frac{x}{\log x}\right) . \tag{2}$$

(2) sharpens a a result of Feinleib [10] and the error term in (2) is best possible.

I proved [7] that if $\varepsilon \to 0$ then (γ is Euler's constant)

$$F(x; 1, 1, + \varepsilon) = (1 + o(1))c^{-\gamma} x/\log \frac{1}{\varepsilon} \tag{3}$$

and (3) of course implies that (1) if true is best possible. Thus to prove our Theorem we only have to prove (1). The proof of (1) will be similar to the one I used in estimating the number of primitive abundant numbers not exceeding x [8].

First I explain the need for the assumption $x > t$. If $a <$

$\sigma(n)/n < a + 1/t$, $n \leqq x$ and t is very large then clearly (1) can not hold since $1 \leqq F(x; a, a + 1/t)$ is greater than $c_1 x/\log t$.

As far as I know it has never been proved that for a suitable α the number of solutions of $\sigma(n)/n = \alpha$ is infinite — or even unbounded in α. It follows by a method of Hornfeck and Wirsing [11] that the number of solutions of $\sigma(n)/n = \alpha$, $n \leqq x$ is $o(x^\varepsilon)$ for every $\varepsilon > 0$ uniformly in α.

To prove (1) denote by $B(x, t)$ the set of integers

$$(4) \qquad 1 \leqq b_1 < \cdots < b_k \leqq x,\ a \leqq \frac{\sigma(b_i)}{b_i} < a + \frac{1}{t} .$$

We have to show that for $x > t$

$$(5) \qquad k < c_1 x/\log t .$$

To prove (5) we show that if we neglect $o(x/\log t)$ of the integers b we can assume that the b's have various properties which make the estimation of their number easier.

First of all we can assume that no b is divisible by a power of a prime p^α, $\alpha > 1$ which is greater than $(\log t)^2$. This is clear since the number of such integers $\leqq x$ is less than

$$(6) \qquad \sum_{\substack{p^\alpha > (\log t)^2 \\ \alpha > 1}} \frac{x}{p^2} < c_2 x/\log t .$$

Write now

$$(7) \qquad b_i = u_i v_i w_i$$

where all prime factors of u_i are $< \log t$, all prime factors of v_i are in $(\log t, t^{1/2})$ and all prime factors of w_i are $\geqq t^{1/2}$.

Now we show that we can assume

$$(8) \qquad u_i < t^{1/10} .$$

For if (8) does not hold then u_i must have at least r distinct prime factors $< \log t$ where $(\log t)^r > t^{1/10}$ or $r > \log t/20 \log\log t$. Thus by a simple computation the number of b's not satisfying (8) is less than

$$(9) \qquad x\Big(\sum_{p < \log t} \frac{1}{P}\Big)\frac{r}{r!} < x\,\frac{(2 \log\log t)^r}{r!} < \frac{c_3 x}{\log t} .$$

Now we consider the b's with $v_j > 1$, i.e., we consider the b's which have at least one prime factor in $(\log t, t^{1/2})$. Let $p_i \mid b_i$ be such a prime factor, then we must have $p_i^2 \nmid b_i$. Now we show that the integers b_i/p_i are all distinct, thus the number of these b's is less than $x/\log t$.

To see this assume $b_i/p_i = b_j/p_j$, $p_j > p_i$. But then

$$(10) \qquad \frac{\sigma(b_i/p_i)}{b_i/p_i} = \frac{\sigma(b_j/p_j)}{b_j/p_j} \quad \text{or} \quad \frac{\sigma(b_i)b_j}{b_i\sigma(b_j)} = \frac{(p_i+1)p_j}{p_i(p_j+1)} .$$

But $a \leqq \sigma(b)/b < a + 1/t$, $p_i < t^{1/2}$, $p_j < t^{1/2}$. Thus

$$(11) \qquad 1 \leqq \frac{\sigma(b_i)b_j}{b_i\sigma(b_j)} < 1 + \frac{1}{at} \quad \text{and} \quad \frac{(p_i+1)p_j}{p_i(p_j+1)} \geqq 1 + \frac{1}{t}$$

(10) and (11) clearly contradict each other. Thus we can henceforth assume that our b's have no prime factor in $(\log t, t^{1/2})$. Thus finally we can restrict ourselves to the b's of the form

$$b_i = u_i w_i$$

where all prime factors of u_i are $< \log t$ and $u_i < t^{1/10}$ and all prime factors of $w_i \geqq t^{1/2}$.

Next we show that we can restrict ourselves to the b's for which

$$(12) \qquad \frac{\sigma(w_i)}{w_i} < 1 + \frac{10}{t^{1/2}} .$$

Consider first the b's which for some $r = 0, 1, \cdots$ have two or more prime factors in $(2^n t^{1/2}, 2^{n+1}t^{1/2})$. The number of these b's is clearly less than (in Σ_r the summation is extended over the primes in $(2^r t^{1/2}, 2^{r+1}t^{1/2})$)

$$x \sum_{r=0}^{\infty} \left(\sum_r \frac{1}{p} \right)^2 < x \sum_{r=0}^{\infty} \frac{1}{(\log 2^r t^{1/2})^2} = x \sum_{r=0}^{\infty} \frac{1}{(r \log 2 + \log t)^2} < \frac{c_1 x}{\log t} .$$

For the b's which have only one prime factor in $(2^r t^{1/2}, 2^{r+1}t^{1/2})$, $r = 0, 1, \cdots$ we evidently have

$$\frac{\sigma(w_i)}{w_i} < \prod_{r=0}^{\infty} \left(1 + \frac{1}{2^r t^{1/2}}\right) < 1 + \frac{10}{t^{1/2}}$$

for $t > t_0$. Thus henceforth we can assume that (12) holds.

Thus we obtained that if we neglect $cx/\log t$ integers than all our integers $b_i < x$ satisfying

$$a \leqq \frac{\sigma(b_i)}{b_i} < a + \frac{1}{t}$$

have the following properties. All their prime factors p^α, $\alpha > 1$ satisfy $p^\alpha < (\log t)^2$, they have no factor in $(\log t, t^{1/2})$ and if we put $b_i = u_i w_i$ where all prime factors of u_i are $\leqq \log t$ then $u_i < t^{1/10}$ and

$$\frac{\sigma(w_i)}{w_i} < 1 + \frac{10}{t^{1/2}} .$$

Now observe that for all the b's which remain we must have constant value $\sigma(u_i)/u_i = \alpha$. To see this assume that, say, $\sigma(u_1)/u_1 > \sigma(u_2)/u_2$ then we have

$$\frac{\sigma(u_1)}{u_1} - \frac{\sigma(u_2)}{u_2} \geqq \frac{1}{u_1 u_2} > \frac{1}{t^{1/5}} \tag{13}$$

or by (13)

$$\frac{\sigma(u_2)}{u_2} < a + \frac{1}{t} - \frac{1}{t^{1/5}} \tag{14}$$

but then by (12) and (14) for $t > t_0$

$$\frac{\sigma(b_2)}{b_2} < \left(a + \frac{1}{t} - \frac{1}{t^{1/5}}\right)\left(1 + \frac{10}{t^{1/2}}\right) < a$$

an evident contradiction.

In view of what we just proved all the b's (neglecting perhaps $cx/\log t$ of them) are of the form

$$u_i w_i, \frac{\sigma(u_i)}{u_i} = \alpha_1, \quad u_i < t^{1/2},$$

where all prime factors of u_i are $\leqq \log t$ and all prime factors of w_i are $\geqq t^{1/2}$.

In a previous paper [9] I proved that there is an absolute constant C so that

$$\sum_{\sigma(u)/u = \alpha} \frac{1}{u} \leqq C. \tag{15}$$

In fact with more trouble we can show $C = 1$ [7], [9].

Now we can complete the estimation of the number of b's not exceeding x.

For fixed u_i the number of w_i for which $u_i w_i$ can be a b is less than the number of integers $\leqq x/u_i$ all whose prime factors are $\geqq t^{1/2}$.

Thus by Brun's method that number is less than

$$\frac{cx}{u_i \log t}$$

summing for u_i we obtain our statement from (15). The restriction $t > t_0$ is clearly irrelevant.

By somewhat more trouble we could prove

$$F\left(x; a, a + \frac{1}{t}\right) \leqq (1 + o(1))F\left(x; 1, 1 + \frac{1}{t}\right) = (1 + o(1))e^{-\gamma}x/\log t .$$

$F(x; a, a + 1/t) \leqq F(x; 1, 1 + 1/t)$ is easily seen to be false in general but for fixed a

$$\lim_{x=\infty} \frac{F(x; a, a + \alpha)}{F(x; 1, 1 + \alpha)} < 1$$

can be proved by the methods of this paper, or $g(a + \alpha) - g(a) < g(1 + \alpha)$.

To see that

$$F\left(x; a, a + \frac{1}{t}\right) \leqq F\left(x; 1, 1 + \frac{1}{t}\right)$$

fails choose $t = 1$ and let $a < 1 + 1/x$. There is no $\sigma(n)/n, n < x$, in $(1, a)$. On the other hand, the perfect numbers 6, 28 etc. are counted in $F(x; a, a + 1)$ but not in $F(x, 1, 2)$. The reader may with justice consider this counterexample as dishonest and in fact by the methods of this paper we can prove

$$F\left(x; a, a + \frac{1}{t}\right) < F\left(x; 1, 1 + \frac{1}{t}\right)$$

if $a > 1 + 2/x$ but we supress the details.

REFERENCES

1. F. Behrend, *Über numeri abundantes*, Preuss. Akad. Wiss. Sitrzungsber, (1932), 322-328 and (1933), 280-293. See also C. R. Wall, P. L. Crass, and D. B. Johnson, *Density bounds for the sum of divisors function*, Math. of Computation, **26** (1972), 113-114.
2. H. Davenport, *Über numeri abundantes*, ibid., (1934), 830-834.
3. H. G. Diamond, *The distribution of values of Euler's Phi function*, Proc. Symp. Pure Math., Amer. Math. Soc. **XXIV**, (1972), 63-75.
4. For the relevant results see e.g. P. Erdös, *On additive arithmetical functions and applications to number theory*, Proc. International Congress Math., Amsterdam, (1954), Vol. **IV**, 13-19, Nordhoff Groningen, 1956 and see also M. Kac, *Probability methods in some problems of analysis and number theory*, Bull. Amer. Math. Soc., **55** (1949), 641-655, and the book of Kubilius, *Probabilistic methods in the theory of numbers*, Amer. Math. Soc. Translations, Vol. 11.
5. P. Erdös, *On the smoothness of the asymptotic distribution of additive arithmetical functions*, Amer. J. Math., **61** (1939), 722-725.
6. ———, *On amicable numbers*, Publ. Math. Debl., **4** (1955-56), 108-111.
7. ———, *Some remarks about additive and multiplicative functions*, Bull. Amer. Math. Soc., **52** (1946), 527-537, see Theorem 3, p. 529.
8. ———, *On primitive abundant numbers*, J, London Math. Soc., **10** (1935), 49-58, see also *On primitive α-abundant numbers*, Acta Arith., **5** (1958), 25-33.
9. ———, *Remarks on number theory I and II*, Acta Arith., **5** (1959), 23-33 and 171-177.
10. A. S. Fainleib, *Distribution of values of Euler's function*, Mat. Zametki, **1** (1967), 645-652 = Math. Notes, 1 (1967), 428-432.
11. B. Hornfeck and E. Wirsing, *Über dia Häufigkeit vollkommener Zahlen*, Math. Annalen, **133** (1957), 431-438, see also E. Wirsing, *Bemerkungen zu der Arbeit über vollkommene Zahlen*, ibid., **139** (1959), 316-318.

12. I. Schoenberg, *Über die asymptotische verteilung reeller Zahlen mod* 1, Math. Z., **28** (1928), 171-199.
13. M. M. Tjan, *On the question of the distribution of values of the Euler function* $\varphi(n)$, Litovsk. Mat. Sb., **6** (1966), 105-119.

Received January 30, 1973.

UNIVERSITY OF WISCONSIN
AND
UNIVERSITY OF NORTHERN ILLINOIS