On the values of Euler's φ-function

by

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Introduction. Let M denote the set of distinct values of Euler's φ-function, that is, $m \in M$ if and only if

$$m = \varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

for some positive integer n. Let $m_1, m_2, m_3, \ldots$ be the elements of M arranged in increasing sequence.

Our main object in this paper is to estimate the sum

$$V(x) = \sum_{m_i \leqslant x} 1$$

from above. Note that $V(x) \geqslant \pi(x)$, for M includes the sequence $\{p-1\}$, and it was shown by Erdös [1] that for each positive ε,

$$V(x) = O\left(\frac{x}{\log^{1-\varepsilon} x}\right).$$

We prove the following

THEOREM. *For each $B > 2\sqrt{2/\log 2}$, we have that*

$$V(x) = O\left(\pi(x) \exp\{B\sqrt{\log\log x}\}\right).$$

We have not yet found a comparable estimate from below; we remark that it may be shown that

$$V(x) = \Omega\left(\pi(x)\,(\log\log x)^t\right)$$

for every fixed t, and we hope to study this further perhaps in a later paper.

An interesting problem is to investigate the gaps in the sequence $\{m_i\}$. Since this includes the sequence $\{p-1\}$, we have that

$$m_{i+1} - m_i = O(m_i^a)$$

for every $a > 3/5$ by Montgomery's estimate [2] for the difference between consecutive primes. It is clear that our theorem gives

$$m_{i+1} - m_i = \Omega\left(\frac{\log m_i}{\exp\{B\sqrt{\log\log m_i}\}}\right)$$

for every $B > 2\sqrt{2/\log 2}$, and it is possible that in fact

$$m_{i+1} - m_i = \Omega(\log m_i),$$

although we cannot prove this. We now give the proof of our main result.

LEMMA 1. *Let $\omega(n)$ denote the number of prime factors of n counted according to multiplicity. Then the number of integers $n \leqslant x$ for which*

$$\omega(n) \geqslant \frac{2}{\log 2} \log\log x$$

is $O\big(\pi(x)\log\log x\big)$.

Proof. Let $\omega'(n)$ denote the number of odd prime factors of n, and $\nu(n)$ the number of distinct prime factors. Then for all y,

$$(1+y)^{\omega'(n)} = \sum_{d|n}{}' y^{\nu(d)}(1+y)^{\omega(d)-\nu(d)}$$

where $\sum'$ denotes a sum restricted to odd d. Hence for real, non-negative y,

$$\sum_{n \leqslant x} (1+y)^{\omega'(n)} \leqslant x \sum_{d \leqslant x}{}' \frac{y^{\nu(d)}}{d}(1+y)^{\omega(d)-\nu(d)} \leqslant x \prod_{3 \leqslant p \leqslant x}\left(1 + \frac{y}{p-1-y}\right),$$

provided $y < 2$. This does not exceed

$$x(\log x)^y \exp\left(\frac{A}{2-y}\right)$$

where A is an absolute constant. Setting $y = t-1$ we have that

$$\sum_{n \leqslant x} t^{\omega'(n)} \leqslant x(\log x)^{t-1}\exp\left(\frac{A}{3-t}\right)$$

provided $1 \leqslant t < 3$, and we deduce that for this range of values of t,

$$\sum_{\substack{n \leqslant x \\ \omega'(n) \geqslant t\log\log x}} 1 \leqslant x(\log x)^{t-1-t\log t}\exp\left(\frac{A}{3-t}\right).$$

Next, set $u = 2/\log 2 < 3$. If $\omega(n) \geqslant u\log\log x$ and $2^k \| n$, we must have $\omega'(n) \geqslant u\log\log x - k$. The number of integers $n \leqslant x$ for which

$k \geqslant \frac{1}{2}u\log\log x$ is $O(x/\log x)$ and so

$$\sum_{\substack{n\leqslant x\\ \omega(n)\geqslant u\log\log x}} 1 \leqslant \sum_{0\leqslant k\leqslant \frac{1}{2}u\log\log x} \sum_{\substack{m\leqslant x/2^k\\ \omega'(m)\geqslant u\log\log x-k}} 1 + O\left(\frac{x}{\log x}\right).$$

Set $k = h\log\log x$ so that h varies in the range $[0, \frac{1}{2}u]$. Certainly $1 \leqslant u-h < 3$, and so the inner sum on the right is

$$\ll \pi(x)\,(\log x)^{(u-h)-(u-h)\log(u-h)-h\log 2} \ll \pi(x)$$

since the maximum value of the exponent of $\log x$ is zero. Summing over $k \leqslant u\log\log x$ we obtain our result.

LEMMA 2. *The number of integers $n \leqslant x$ which have no prime factor exceeding*

$$x^{1/6\log\log x}$$

is

$$O\big(\pi(x)\log\log x\big).$$

Proof. We divide the integers $n \leqslant x$ into two classes. If $n \leqslant \sqrt{x}$ or $\omega(n) \geqslant u\log\log x$, n belongs to the first class. Otherwise it belongs to the second class.

By Lemma 1, the number of integers in the first class is $O\big(\pi(x)\log\log x\big)$. If n belongs to the second class, its largest prime factor p must satisfy

$$p^{u\log\log x} > \sqrt{x}.$$

Since $u < 3$ this gives the result.

Proof of the Theorem. There exists an absolute constant c such that for all $n \geqslant 1$,

$$n/\varphi(n) \leqslant c\log\log 3\varphi(n).$$

Let $l = c\log\log 3x$, so that if $\varphi(n) \leqslant x$, then $n \leqslant xl$.

Let m be a value of φ not exceeding x. Either $\omega(m) \geqslant u\log\log x$, or $m = \varphi(n)$ where $n \leqslant xl$ and $\omega\{\varphi(n)\} < u\log\log x$. Therefore

$$V(x) \leqslant \sum_{\substack{m\leqslant x\\ \omega(m)\geqslant u\log\log x}} 1 + \sum_{\substack{n\leqslant xl\\ \omega\{\varphi(n)\}<u\log\log x}} 1.$$

The first sum is $O\big(\pi(x)\log\log x\big)$ by Lemma 1, and it remains to study the second. Note that $l \geqslant 1$ for $x \geqslant 1$, moreover that for $x > e^e$, which we may assume, the function

$$x^{1/6\log\log x}$$

is increasing. We may therefore restrict our attention to those n in the second sum with at least one prime factor larger than this; by Lemma 2 the number of integers $n \leqslant xl$ not counted is

$$O\big(\pi(x)\,(\log\log x)^2\big).$$

In the remaining sum, we may write $n = mp$ where

$$p > x^{1/6\log\log x}, \quad m < lx^{1-1/6\log\log x}.$$

Then

$$V(x) \leqslant \sum_{\omega\{\varphi(m)\}<u\log\log x} \pi\left(\frac{xl}{m}\right) + O\big(\pi(x)\,(\log\log x)^2\big)$$

$$\ll \frac{x(\log\log x)^2}{\log x} \sum_{\omega\{\varphi(m)\}<u\log\log x} \frac{1}{m}.$$

We do not restrict the size of m in this sum, as the series is convergent, as we will show.

Consider the function

$$f(z) = \sum_{n=1}^{\infty} \frac{z^{\omega\{\varphi(n)\}}}{n}.$$

We are only concerned with real z in the range $0 \leqslant z < 1$, and we show that for these values of z the series is convergent. Incidentally, it is therefore absolutely convergent, and so $f(z)$ is well-defined, for $|z| < 1$. The behaviour of this series on the unit circle $|z| = 1$ is an interesting and complicated problem.

Since $\omega\{\varphi(n)\}$ is additive $z^{\omega\{\varphi(n)\}}$ is multiplicative and

$$f(z) = \prod_p \left(1 + \frac{z^{\omega(p-1)}}{p-z}\right) \leqslant \exp \sum_p \frac{z^{\omega(p-1)}}{p-z}$$

for $0 \leqslant z < 1$, provided the series on the right converges.

We apply the following result of Erdös [1]. For every $\varepsilon > 0$ there exists a positive $\delta = \delta(\varepsilon)$ such that the number of primes $p \leqslant x$ for which

$$|\nu(p-1) - \log\log x| \geqslant \varepsilon \log\log x$$

is

$$O\left(\frac{x}{(\log x)^{1+\delta}}\right).$$

Let k and H be positive numbers. Then

$$\sum_{\nu(p-1)\leqslant k} \frac{1}{p} \leqslant \sum_{p\leqslant H} \frac{1}{p} + \int_H^{\infty} \frac{1}{t^2} \Big(\sum_{\substack{p\leqslant t \\ \nu(p-1)\leqslant k}} 1 \Big)\, dt.$$

We select

$$H = H(k) = \exp\exp\left(\frac{k}{1-\varepsilon}\right)$$

so that in the integrand, the condition $\nu(p-1) \leqslant k$ implies that

$$\nu(p-1) \leqslant (1-\varepsilon)\log\log t.$$

The integral is therefore convergent, and we have that for $\varepsilon > 0$,

$$\sum_{\nu(p-1)\leqslant k} \frac{1}{p} \leqslant \frac{k}{1-\varepsilon} + C(\varepsilon)$$

where $C(\varepsilon)$ is independent of k. Therefore for $0 \leqslant z < 1$,

$$\sum_p \frac{z^{\nu(p-1)}}{p} = \sum_{k=0}^{\infty} z^k \sum_{\nu(p-1)=k} \frac{1}{p} = (1-z)\sum_{k=0}^{\infty} z^k \sum_{\nu(p-1)\leqslant k} \frac{1}{p}$$

$$\leqslant (1-z)\sum_{k=0}^{\infty}\left(\frac{kz^k}{1-\varepsilon} + C(\varepsilon)z^k\right) \leqslant \frac{z}{(1-\varepsilon)(1-z)} + C(\varepsilon).$$

Since $\omega(p-1) \geqslant \nu(p-1)$, this gives

$$\sum_p \frac{z^{\omega(p-1)}}{p-z} \leqslant \sum_p \frac{z^{\nu(p-1)}}{p} + \sum_p \frac{1}{p(p-1)} \leqslant \frac{z}{(1-\varepsilon)(1-z)} + C'(\varepsilon),$$

and so

$$f(z) \leqslant C''(\varepsilon)\exp\left\{\frac{z}{(1-\varepsilon)(1-z)}\right\}, \quad \text{for} \quad 0 \leqslant z < 1,$$

where $C'(\varepsilon)$ and $C''(\varepsilon)$ depend on ε only. We are now ready to estimate the sum

$$\sum_{\omega\{\varphi(m)\}<u\log\log x} \frac{1}{m}.$$

For $z < 1$, this does not exceed

$$f(z)z^{-u\log\log x}.$$

We may choose z optimally, and we select the value which gives

$$\left(\frac{z}{1-z}\right)^2 = (1-\varepsilon)u\log\log x.$$

Therefore

$$\sum_{\omega\{\varphi(m)\}<u\log\log x} \frac{1}{m} \leqslant C''(\varepsilon)\exp\left\{2\sqrt{\frac{u\log\log x}{1-\varepsilon}}\right\}$$

and so for every $B > 2\sqrt{2/\log 2}$, we have that

$$V(x) = O\left(\pi(x)\exp\{B\sqrt{\log\log x}\}\right).$$

This completes the proof.

References

[1] P. Erdös, *On the normal number of prime factors of p − 1 and some related problems concerning Euler's φ-function*, Quart. Journ. Math. 6 (1935), pp. 205–213.

[2] H. L. Montgomery, *Zeros of L-functions*, Invent. Math. 8 (1969), pp. 346–354.

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