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PROBLEMS AND RESULTS IN FINITE AND INFINITE COMBINATORIAL ANALYSIS

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In this paper we investigate edge decomposition and imbedding of graphs. In a previous paper (1) we investigated vertex and edge decompositions of graphs, we solved nearly all the problems on vertex decomposition but left most of the problems on edge decompositions open. First we introduce some notation, we found it convenient to use slightly different notations in (1). $\alpha(g)$ denotes the cardinal number of the vertices of g. K_α is the complete graph of α vertices, $\beta(g)$ is the smallest cardinal number for which g does not contain a K_β. C_n is a circuit of n vertices. Let g be a graph. A sequence g_ξ, $o \leq \xi < \gamma$ is said to be an edge decomposition of type γ if every edge of g belongs to one and only one of the g_ξ's.

$(g_1, \alpha) \longrightarrow (g_2, \gamma)$ means that every g with $\alpha(g) = \alpha$ which does not contain a subgraph isomorphic to g_1 has an edge decomposition λ of type γ where no g_ξ contains a subgraph isomorphic to g_2. In (1) we usually assumed $g_1 = K_\beta$, $g_2 = K_\delta$.

$g \longrightarrow (g_\xi)^2_\gamma$ means that for every edge decomposition g_α, $o \leq \alpha < \gamma$ of g of type γ for at least one $\alpha < \gamma$, g_α contains a subgraph isomorphic g_ξ. Thus $K_\omega \longrightarrow (K_\omega, K_\omega)^2$ is a restatement of the well known theorem of Ramsey.

In (1) we proved (among others) the following theorem. Assume G.C.H. (generalized hypothesis of the continuum) then

$(\rho \geq o)$ $(K_\omega, \omega_{\rho+4}) \not\longrightarrow (K_n, \omega_\rho)$ for every $n < \omega$.

We could not decide whether $(K_\omega, \omega_i) \longrightarrow (K_n, \omega)$ is true for $i = 2$ or $i = 3$ and for some $3 \leq n < \omega$. $(K_\omega, \omega_1) \longrightarrow (K_n, \omega)$ is true for every $3 \leq n < \omega$ since according to an old result of Erdös and Tukey K_{ω_1} is the union of ω trees.

One gets very interesting unsolved problems for edge decomposition of finite graphs. Denote by $f(k, \ell, r)$, $k \geq \ell \geq 4$, $r \geq 2$, the smallest integer (if it exists) for which $(K_k, f(k, \ell, r)) \not\longrightarrow (K_\ell, r)$. In other words $f(k, \ell, r)$ is the smallest integer for which there is a graph of $f(k, \ell, r)$ vertices which does not contain a K_k but if we color its edges by r colors at least one of the colors contains a $K_{\ell - 1}$. We conjectured that $f(k, \ell, r)$ exists for every $k \geq \ell \geq 4$, $r \geq 2$.

$f(7, 4, 2) = 6$ is well known and easy to see. Tosa proved that $f(5, 4, 2)$ exists (1) but he did not determine its value. Graham (7) proved $f(6, 4, 2) = 8$. Finally Folkman (6) proved that $f(k, k, 2)$ exists for every $k \geq 4$, but his proof probably gives a very poor upper bound for $f(k, k, 2)$. Folkman probably had a proof that $f(k, k, 3)$ exists. The general case is open.

The principal lemma of Folkman is the following theorem which is of great interest in itself: For each positive integer n and each finite graph G there is a finite graph $H(N, G)$ satisfying $\beta(G) = \beta(H(N, G))$ and if the vertices of G are partitioned into n disjoint sets $C_1, \ldots, C_n$ then for some $i, 1 \leq i \leq n$ there is a set $S \subset C_i$ such that the subgraph of $H(N, G)$ spanned by S is isomorphic to G.

It would be interesting to try to extend this theorem to infinite graphs G and when n can be any infinite cardinal number.

the proof of Folkman is by induction with respect to n and probably would have to be modified to work in this case.

It is quite possible that for every infinite cardinal number $f(k, k, m) = 2^{2^m}$.

Before we leave this subject we make the following remark: Assume that there is a $\lambda \longrightarrow (\lambda, 3)^2$ but $\lambda \not\longrightarrow (\lambda, 4)^2$ ($\longrightarrow$ here denotes the usual arrow symbol of Erdös and Rado (2) and λ is any ordinal or order type). Let g be the graph whose vertices form a set of type λ and which does not contain a K_4 and which does not contain an independent set of type λ. Using $\lambda \longrightarrow (\lambda, 3)^2$ we obtain by a simple argument (the details of which we leave to the reader) that if we color the edges of g by n colors ($n < \omega$) then one of the colors contains a triangle. It then easily follows by well known arguments that g has a finite subgraph with the same property, or $f(4, 4, n)$ exists for every n.

The only trouble with this argument is that it is quite possible that such a λ does not exist. Recently Chang proved $\omega^\omega \longrightarrow (\omega^\omega, 3)^2$. $\omega^\omega \longrightarrow (\omega^\omega, 4)^2$ is still open but Chang believes that it probably holds.

Galvin asked us in a letter if

$$(C_3, n) \not\longrightarrow (C_5, 2) \tag{1}$$

holds for some n. We showed that for every k and ℓ there is a smallest integer $g(k, \ell)$ such that

$$(C_{2k-1}, g(k, \ell)) \not\longrightarrow (C_{2k+1}, \ell) \tag{2}$$

but we did not even determine $g(2, 2)$. We will outline the proof of (1) and (2) later.

Galvin also raised the following interesting question: A class of graphs has the G-R (Galvin-Ramsey (this terminology is due to us)) property if for every graph g_1 of the class there is a graph g_2 of the class so that $g_2 \longrightarrow (g_1, g_1)$. The class has the unrestricted G-R property if for every g_1 of the class and to every γ there is a g_2 of the class so that $g_2 \longrightarrow (g_1)_\gamma$. ($g_2 \longrightarrow (g_1)_\gamma$ is a shorthand notation for $g_2 \longrightarrow (g_1, g_1 ---)$, written γ times in the bracket). Ramsey's theorem and its generalizations show that the class of all finite graphs has the G-R property and the class of all graphs has the unrestricted G-R property. Galvin in particular asked whether the class of all finite graphs not containing a triangle has the G-R property, (2). We cannot at present answer this question, but we will give some classes of graphs which have the G-R property.

First of all it is known that the class of all finite bipartite graphs have the G-R property and the class of all bipartite graphs has the unrestricted G-R property. Trivially the star has the G-R property.

To get new non-trivial classes of graphs having the G-R property we define some graphs which we considered in previous papers (3). Let λ be an ordinal number, the graph g_λ is defined as follows: The vertices of g_λ are the pairs of ordinals (α, β), $0 \leq \alpha < \beta < \lambda$. Two vertices (α, β) and (γ, δ) are joined if $\alpha < \beta = \gamma < \delta$. Clearly g_λ contains no triangles but it contains a pentagon for every $\lambda \geq 4$. First we show

(3) $$g_\omega \longrightarrow (g_\omega, g_\omega).$$

To prove (3) observe that the splitting of the edges of g_ω into two classes corresponds to the splitting of the triangles (a, b, c), $a < b < c < \omega$ into two classes. Ramsey's theorem now implies that there is an infinite sequence $a_1 < \cdots$ all whose triplets are in the same class. This proves (3). The same proof gives that for every k there is an n such that

(4) $$g_n \longrightarrow (g_k, g_k).$$

(3) and (4) immediately implies that the class of finite graphs (and the class of all graphs) imbeddable in g_ω have property G-R. Also (4) immediately implies (1).

Assuming G.C.H. the above proof gives, using the results of (4), that for every β and γ there is an α such that

(5) $$g_\alpha \longrightarrow (g_\beta)_\gamma.$$

(5) clearly implies that the class of graphs which can be imbedded into some g_α has the unrestricted G-R property.

Another theorem of (4) states

(6) $$\omega_{\alpha+3} \longrightarrow (\omega)^3_{\omega_\alpha}$$

and from (6) we obtain $g_{\omega_{\alpha+3}} \longrightarrow (g_\omega)_{\omega_\alpha}$. Thus the same argument which we used in the proof (1) gives

(7) $$(C_3, \omega_{\alpha+3}) \not\longrightarrow (C_5, \omega_\alpha)$$

We cannot decide whether

(8) $$(C_3, \omega_{\alpha+2}) \not\longrightarrow (C_5, \omega_\alpha)$$

holds. In fact (8) is open even for $\alpha = 0$. In (4) we showed

$$\omega_{\alpha+2} \not\longrightarrow (4)^3_{\omega_\alpha}$$

thus (8) cannot be proved by this method.

Define the graphs S_η (introduced by Specker (7)) as follows: The vertices of S_η are the triplets of ordinals

(α, β, γ), $\alpha < \beta < \gamma < \eta_*(\alpha, \beta, \gamma)$ and $(\alpha', \beta', \gamma')$ are joined if $\beta < \alpha' < \gamma < \beta'$. It is easy to see that S_η contains no triangles and the method which proved (3) gives $S_\omega \longrightarrow (S_\omega, S_\omega)$. Thus the class of the finite graphs (and the class of all graphs) imbeddable in S_ω has the G-R property and the class of graphs imbeddable in S_η for some η has the unrestricted G-R property.

In (3) we considered various graphs related to g_α and S_α. In this way we obtain several other classes of graphs which have the G-R property, respectively the unrestricted G-R property, also with the help of these graphs we can prove (2). We supress the details.

It does not seem easy to characterise those finite graphs which can be imbedded into S_ω or g_ω. It is easy to see that if g can be imbedded in S_ω or g_ω we must have $\beta(g) = 3$, but not every finite graph with $\beta(g) = 3$ (i.e. not containing a triangle) can be imbedded into S_ω or g_ω. The simplest graph with $\beta(g) = 3$ and not imbeddable in g_ω has the vertices $x_1, x_2, x_3; y_1, y_2, y_3; z$. z is joined to x_3 and y_3, x_3 and y_3 are not joined but all other 8 edges (x_i, y_j) occur in our graph. This graph can easily be imbedded in S_ω. To show that there are finite graphs with $\beta(g) = 3$ which cannot be imbedded in S_ω we observe that it is not hard to show that the chromatic number of every subgraph g of S_ω with $\alpha(g) \leq n$ is less than $(\log n)^\epsilon$ and it is known (5) that there are graphs with $\beta(g) = 3$, $\alpha(g) = n$ whose chromatic number is greater than $n^{\frac{1}{2}-\epsilon}$. We have not determined the smallest $\alpha(g)$ with $\beta(g) = 3$, not imbeddable in S_ω. Finally it is easy to see that g_ω can be imbedded into S_ω.

It would be interesting to find a graph g satisfying

(9) $$\beta(g) = 3, \quad \alpha(g) = \omega$$

so that every graph which satisfies (9) can be imbedded into g. It is not clear if such a g exists. It is easy to see that there is a g satisfying (9) into which every g with $\beta(g) = 3, \alpha(g) < \omega$ can be imbedded, it would be very useful to have a simple construction for such a graph, it seems doubtful if this is possible.

Define the graph $T_r(\alpha)$ as follows: The vertices of $T_r(\alpha)$ are r-tuples of ordinals $(t_1, \cdots, t_r)$, $o \leq t_i < \alpha$, $1 \leq i \leq r$. $(t_1, \cdots, t_r)$ and $(t_1', \cdots, t_r')$ are joined if there are two indices i and j, $1 \leq i \leq r$; $1 \leq j \leq r$ so that $t_i < t_i'$ and $t_j' > t_j$. It easily follows from Ramsey's theorem that $\beta(T_r(\alpha)) = \omega$. $T_2(\alpha)$ has been extensively studied in (1). It is not hard to prove that the graphs imbeddable in some $T_r(\alpha)$ (r fixed α variable) have the unrestricted G-R property. At first we thought that perhaps every graph g with $\beta(g) = \omega$ can be imbedded into some $T_2(\alpha)$, but Galvin showed that g_ω cannot be imbedded into any $T_r(\alpha)$. Assume that g_ω could be imbedded into $T_r(\alpha)$. This would imply that the edges of g_ω are split into $r(r-1)$ classes where two edges belong to the same class if in the imbedding the same pair (i, j) (in $t_i < t_i'$ $t_j' > t_j$) corresponds to them. From (3) we obtain that there is a whole g_ω all whose edges are in the same class and this would imply an infinite descending sequence of ordinal numbers - an evident contradiction.

We do not know whether all the graphs which can be imbedded into some $T_r(\alpha)$ can also be imbedded into some $T_2(\alpha')$. Also we do not know if the class of graphs with $\beta(g)^2 = \omega$ form a G-R class or an unrestricted G-R class.

Let g be any graph, x a vertex of g, y_1, y_2, $\cdots$ the vertices joined to x. $g(x)$ is the subgraph of g spanned by y_1, $\cdots$. We define by transfinite induction for every $\alpha \geqslant 0$ the class of graphs of rank α. The graphs of rank 0 are the finite graphs. The graphs of rank one are the graphs each vertex of which has finite valency. Assume that we have defined for every $\beta < \alpha$ the graphs of rank β. The graphs of rank α are the graphs g such that for every vertex x of g, $g(x)$ has rank $\beta(x) < \alpha$.

We can prove that $\beta(g) = \omega$ if and only if g has rank α for some ordinal α. We supress the details.

It would be interesting to decide whether the class of graphs not containing a rectangle has the G-R property. In (1) it is proved that every graph which contains no rectangle has an edge decomposition into countably many trees, thus this class certainly does not have the unrestricted G-R property. We have not found any non-trivial class of graphs which has the G-R property but does not have the unrestricted G-R property.

Let g_1 be any finite graph. Does the class of graphs which do not have a subgraph isomorphic to g_1 have the G-R property? If the answer is affirmative then for every g_2 which does not contain g_1 as a subgraph and for every sufficiently large n $(g_1, n) \not\longrightarrow (g_2, 2)$.

Many generalizations of all these questions are possible but we leave these for the reader, we just state for the time being one more problem:

It would be of interest to investigate the graphs for which

$$g \longrightarrow (g, g) \tag{10}$$

holds. In this note we found a large class of graphs which satisfy (10). First of all we want to define another graph g, $\alpha(g) = \omega$ which satisfies (10). The vertices of g are $x_1, x_2, \cdots,$ $y_1, y_2, \cdots$. x_i is joined to y_j if $j > i$. It is easy to show that g satisfies (10). It seems very hard to characterize all graphs with $\alpha(g) = \omega$ which satisfy (10). The only connected graph known to us with $\alpha(g) = \omega_1$ which satisfies (10) is the star, perhaps non-trivial graphs satisfying (10) only exist if $\alpha(g)$ is a Ramsey cardinal.*

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*Eric Milner has found a simple positive answer to this problem.

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