[10]

The Minimal Regular Graph Containing a Given Graph

Paul Erdös and Paul Kelly

In the first book on graph theory ever written, König proved the following result. If G is any graph, and d is the maximum degree of the points of G, then it is possible to add new points and to draw new lines joining either two new points or a new point with an old point, so that the result is a regular graph H of degree d.

In this lecture the authors, Paul Erdös and Paul Kelly, determine the smallest number of new points which must be added to G to obtain such a graph H. The result depends only on the degree sequence of the given graph G. A preliminary version of this proof appeared in the American Mathematical Monthly [2]. *The present exposition is more gentle and is liberally illustrated.*

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F.H.

Let G be a graph of order n and maximum degree d. What is the least possible order of a graph H which is regular of degree d and which contains G as an induced subgraph? One can regard the problem in the following way. The graph G is given together with a set I of m new isolated points. A graph H is formed from G and I by adding joins (new lines) between pairs of points in I and between points in I and G, but no joins are added between pairs of points in G. It is desired to make H regular of degree d and to have m as small as possible.

In Fig. 10.1 we illustrate such a completion for each of the three (4, 3) graphs, whose lines are drawn solid, to a minimal regular graph H_i containing G_i as an induced subgraph. The new lines of H_i are drawn dashed.

It is well known that any graph G has a completion H. Suppose that H is constructed and that its order is $m + n$. Let $v_1, v_2, \cdots, v_n$ be the points in G and $u_1, u_2, \cdots, u_m$ those in I. Let F be the subgraph of H induced by I. Denote the degree of v_i as a point of G by d_i, and let $e_i = d - d_i$ denote the *deficiency*

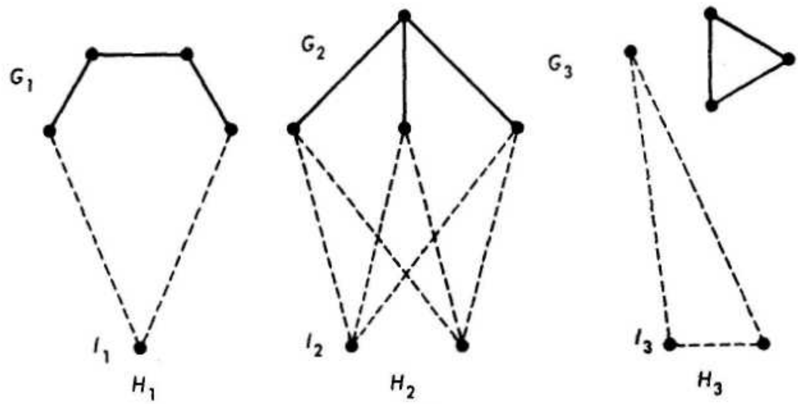


Fig. 10.1.

of v_i, that is, the number of joins needed to complete v_i to degree d. Finally, call the number $s = \Sigma\, e_i$, the *total deficiency*, and $e = \max e_i$, the *maximum deficiency*.

In H there are clearly s lines which join a point of F and a point of G. Since each of the m points in F is adjacent to at most d points of G, it follows that

$$md \geq s\,. \tag{1}$$

The sum of the degrees of $u_1, u_2, \cdots, u_m$ as points of F is $md - s$, and F can have at most $m(m-1)/2$ lines, so that $m(m-1) \geq md - s$ or

$$m^2 - (d+1)m + s \geq 0\,. \tag{2}$$

Clearly m can be no less than the deficiency of any point of G, so that

$$m \geq e\,. \tag{3}$$

Finally, the sum of the degrees in any graph is even; hence

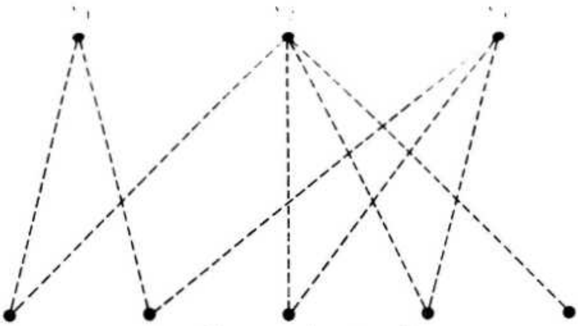
$$(m+n)d \text{ is an even integer.} \tag{4}$$

The conditions (1), (2), (3), and (4) are thus necessary conditions which m must satisfy. We will show that they are also sufficient.

THEOREM. Let G be a graph of order n, maximum degree d, maximum deficiency e, and total deficiency s. Let H be a regular graph of degree d containing G as an induced subgraph. A necessary and sufficient condition that $m + n$ be the least possible order for H is that m be the least integer satisfying (1) $md \geq s$, (2) $m^2 - (d+1)m + s \geq 0$, (3) $m \geq e$, and (4) $(m+n)d$ is even.

To establish the sufficiency, we require a construction. Let m satisfy (1), (2), (3), and (4), and let I consist of $u_1, u_2, \cdots, u_m$, as before. Let $v_1, v_2, \cdots, v_k$

denote the points of G with positive deficiencies. Because $s \geq md - mn$ and $e \leq m$, the points of G can be completed by lines joining G with I. Let the completion be accomplished in the following way, as illustrated in Fig. 10.2. In this example, $m = 5$ and G is a graph in which exactly three points v_1, v_2, and v_3 have positive deficiencies, which are respectively 2, 4, and 3. First, v_1 is completed by joins to $u_1, u_2, \cdots, u_{e_1}$. Then v_2 is completed by joins to successive points u_i, starting with u_{e_1+1} and continuing cyclically, and so on.



$e_1 = 2, \quad e_2 = 4, \quad e_3 = 3.$

Fig. 10.2.

The degrees attained by points of I cannot differ by more than 1 from each other at any stage of this construction. Thus this is also true when all the points of G are complete. Let h and r be the quotient and remainder of s/m, so $s = hm + r$. Thus, now that the points of G have been completed, the first r of the points of I have degree $h + 1$ and the remaining $m - r$ have degree h.

We must still show that there is a graph F, with the m points of I, in which r have degree $d - h - 1$ and the others have degree $d - h$. Suppose $a_i = d - h$ when $i = 1, 2, \cdots, m - r$, and $a_i = d - h - 1$ when $i = m - r + 1, \cdots, m$. By a theorem of Erdös and Gallai [1] applied to this situation, there is such a graph F if $d - h < m$, Σa_i is even, and

$$\sum_{i=1}^{k} a_i \leq \frac{k(k-1)}{2} + \sum_{i=k+1}^{p} \min\{k, a_i\}$$

for all $$k = 1, 2, \cdots, p - 1.$$

Substituting $s = hm + r$ into condition (2), it follows at once that $d - h \leq m - 1 + r/m$; and since $r/m < 1$ while $d - h$ and $m - 1$ are integers, $d - h < m$. Since there are s lines joining points of G and I, $\Sigma a_i = md - s$. Letting q denote as usual the number of lines in G, we find $s = nd - 2q$ so that

$$md - s = md - (nd - 2q) = (m + n)d - 2(nd - q).$$

By (4), $(m + n)d$ is even, so $md - s$ is even. The last of the three conditions for the existence of the graph F is routinely verified. Therefore there is a completion of G to a regular graph H of degree d and order $m + n$, proving the theorem.

Among all graphs of order n, the maximum value of this minimum is n. It is easily seen that since $e \leq d < n$ and $s < nd$, n satisfies the four conditions and hence is an upper bound. That it is the least upper bound follows from an example. Let G be $K_n - x$, the graph obtained from a complete graph of

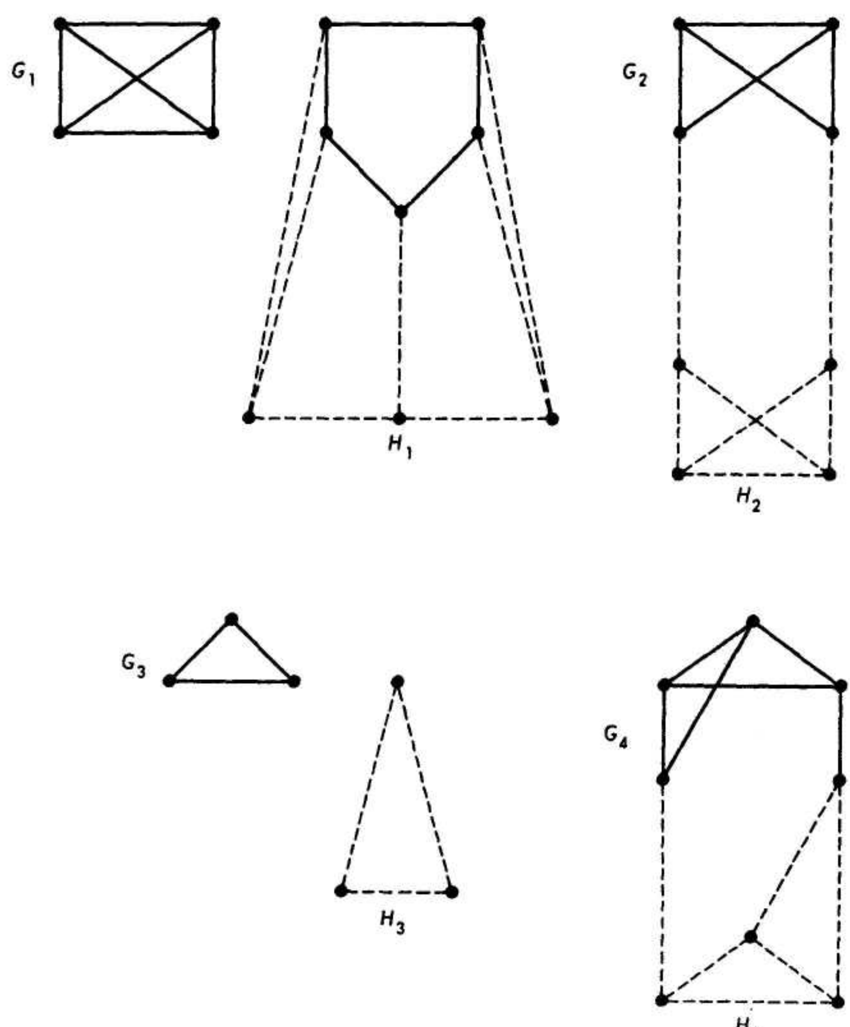


Fig. 10.3.

order n by deleting one line, whence $s = 2$ and $d = n - 1$. Then condition (2) is $m^2 - mn + 2 \geq 0$, which implies $m \geq n$.

We conclude with four examples showing that each of the four conditions can be the one which determines the minimum order m of the completion.

Graph G_1 of Fig. 10.3 has four points of degree 3 and five points of degree 2, so that $n = 9$, $d = 3$, $e = 1$, and $s = 5$. The smallest value of m satisfying (2), (3), and (4) is 1, but this does not satisfy (1). The minimal completion H must have three additional points.

For graph G_2, which is $K_4 - x$, $n = 4$, $d = 3$, $e = 1$ and $s = 2$, and we know that $m = 4$. However, the number 2 satisfies (1), (3), and (4) simultaneously.

In the third graph, consisting of K_3 and an isolated point, $n = 4$, $d = 2$, $e = 2$, and $s = 2$. Whereas the number 1 satisfies (1), (2), and (4), (3) forces m to be 2.

In graph G_4, $n = 5$, $d = 3$, $e = 2$, and $s = 3$. Together (1), (2), and (4) imply that $m \geq 1$ while (1), (2), and (3) imply $m \geq 2$. All four conditions imply $m = 3$.

References

[1] P. Erdös and T. Gallai, Graphen mit Punkten vorgeschriebenen Grades. *Mat. Lapok.* **11**(1960) 264–274.

[2] ——— and P. Kelly, The minimal regular graph containing a given graph. *Amer. Math. Monthly* **70**(1963) 1074–1075.