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SOME REMARKS ON CHROMATIC GRAPHS

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A graph G is said to be *k-chromatic* if its vertices can be split into k classes so that two vertices of the same class are not joined (by an edge) and such a splitting into $k-1$ classes is not possible. The *chromatic number* will be denoted by $H(G)$. A graph is called *complete* if any two of its vertices are joined. Denote by $K(G)$ the number of vertices of the largest complete subgraph of G. The *complementary graph* $\bar{G}$ of G is defined as follows: $\bar{G}$ has the same vertices as G and two vertices are joined in $\bar{G}$ if and only if they are not joined in G. A set of vertices of G is called *independent* if no two of them are joined. $I(G)$ denotes the greatest integer for which there is a set of $I(G)$ independent vertices of G. We evidently have

$$H(G) \geqslant K(G) = I(\bar{G}).$$

Throughout this paper G_n will denote a graph of n vertices, $c_1, c_2, \ldots$ will denote positive absolute constants. Vertices of G will be denoted by $X_1, X_2, \ldots$, $G-X_1-\ldots-X_r$ will denote the graph G from which the vertices $X_1, \ldots, X_r$ and all the edges incident to them have been omitted. $G(X_1, \ldots, X_m)$ denotes the subgraph of G spanned by the vertices $X_1, \ldots, X_m$.

Tutte and Ungar (see [2]) and Zykov [10] were the first to show that for every l there is a graph G with $K(G) = 2$ (i.e. G contains no triangle) and $H(G) = l$. I proved [6] that for every n there is a G_n with $K(G_n) = 2$ and $H(G_n) > cn^{1/2}/\log n$. On the other hand, it easily follows from a result of Szekeres and myself [7] that if $K(G_n) = 2$, then $H(G_n) < c_1 n^{1/2}$.

It is an open and difficult problem to decide if for every n there is G_n with $K(G_n) = 2$ and $H(G_n) > c_2 n^{1/2}$ (**P 573**).

In the present note we prove the following

THEOREM. *For every n there is a G_n satisfying*

$$\frac{H(G_n)}{K(G_n)} > \frac{c_3 n}{(\log n)^2}. \tag{1}$$

But, on the other hand, for every G_n we have

$$\frac{H(G_n)}{K(G_n)} < \frac{c_4 n}{(\log n)^2}. \tag{2}$$

It seems likely that

$$\lim_{n\to\infty}\left(\max_{G_n}\left(\frac{H(G_n)}{K(G_n)}\right)\Big/\frac{n}{(\log n)^2}\right) = C \tag{3}$$

exists (**P 574**), but I have not been able to prove (3). By the methods of this note it would be easy to prove that

$$\frac{(\log 2)^2}{4} \leqslant \liminf_{n\to\infty}\left(\max_{G_n}\left(\frac{H(G_n)}{K(G_n)}\right)\Big/\frac{n}{(\log n)^2}\right)$$
$$\leqslant \limsup_{n\to\infty}\left(\max_{G_n}\left(\frac{H(G_n)}{K(G_n)}\right)\Big/\frac{n}{(\log n)^2}\right) \leqslant (\log 2)^2$$

First we prove (1). It is known [5] that for every $n > n_0$ there is a graph G_n so that

$$K(G_n) \leqslant \frac{2\log n}{\log 2}, \qquad K(\bar{G}_n) \leqslant \frac{2\log n}{\log 2}. \tag{4}$$

From the definition of the chromatic number we immediately obtain that for every graph G_n

$$H(G_n) \geqslant \frac{n}{I(G_n)} = \frac{n}{K(\bar{G}_n)}. \tag{5}$$

The proof of (5) is immediate since the vertices of G_n can be decomposed into $H(G_n)$ independent sets or $n \leqslant H(G_n)I(G_n)$.

(4) and (5) immediately imply (1).

To prove (2) we first prove two simple lemmas.

LEMMA 1. *Let* $\binom{u+v}{v} \geqslant n$. *Then* $uv > c_5(\log n)^2$.

Without loss of generality we can assume $u \geqslant v$. We then have

$$n \leqslant \binom{u+v}{v} \leqslant \binom{2u}{v} \leqslant \frac{(2u)^v}{v!} < \left(\frac{2eu}{v}\right)^v. \tag{6}$$

$uv > c_5(\log n)^2$ follows from (6) by a simple computation.

In fact, with somewhat more trouble we could prove the following stronger result:

If $\binom{u+v}{v} \geqslant n$, then

$$\min\ (uv) = \left[\frac{t}{2}\right]\left[\frac{t+1}{2}\right], \tag{7}$$

where t is the smallest integer for which $\binom{t}{[t/2]} \geqslant n$.

From (7) we obtain by a simple computation

$$uv \geqslant \bigl(1+o(1)\bigr)\left(\frac{\log n}{\log 4}\right)^2.$$

LEMMA 2. *Let $n \geqslant m \geqslant N$. Assume that for every m and every subgraph $G(X_1, \ldots, X_m)$ of G_n we have $I\bigl(G(X_1, \ldots, X_m)\bigr) \geqslant l$. Then*

$$H(G_n) \leqslant \frac{n}{l} + N.$$

Let $X_1^{(1)}, \ldots, X_{n_1}^{(1)}$ be a maximal system of independent vertices of G_n $(n_1 = I(G_n))$. $X_1^{(2)}, \ldots, X_{n_2}^{(2)}$ is a maximal system of independent vertices of $G_n - X_1^{(1)} - \ldots - X_{n_1}^{(1)}$; $X_1^{(3)}, \ldots, X_{n_3}^{(3)}$ — a maximal system of independent vertices of $G_n - X_1^{(1)} - \ldots - X_{n_1}^{(1)} - X_1^{(2)} - \ldots - X_{n_2}^{(2)}$ etc. We continue this process until

$$\sum_{i=1}^{r} n_i > n - N.$$

By our assumption $n_i \geqslant l$ for all $i, 1 \leqslant i \leqslant r$. Thus $r \leqslant n/l$. The $X_j^{(i)}, 1 \leqslant j \leqslant n_i, 1 \leqslant i \leqslant r \leqslant n/l$, are the vertices of the i-th colour and the remaining fewer than N vertices all get different colours. Thus Lemma 2 is proved.

Now we are ready to prove (2). It is known [7] that

$$\binom{K(G_m) + K(\bar{G}_m) - 2}{K(G_m) - 1} \geqslant m. \tag{8}$$

Thus by Lemma 1

$$K(G_m)\,K(\bar{G}_m) > c_5 (\log m)^2. \tag{9}$$

From (9) we infer that if $m \geqslant n/(\log n)^2$, then for every subgraph $G(X_1, \ldots, X_m)$ we have

$$I\bigl(G(X_1, \ldots, X_m)\bigr) > \frac{c_5 (\log m)^2}{K\bigl(G(X_1, \ldots, X_m)\bigr)} > \frac{c_6 (\log n)^2}{K(G_n)}. \tag{10}$$

Now apply Lemma 2 with $N = n/(\log n)^2$, $l = c_6 (\log n)^2 / K(G_n)$. We then obtain

$$H(G_n) < \frac{nK(G_n)}{c_6 (\log n)^2} + \frac{n}{(\log n)^2} \tag{11}$$

and (2) immediately follows from (11). This completes the proof of our theorem.

Finally we state some more problems. Denote by $G(n; m)$ a graph of n vertices and m edges. It is easy to see that if $H\bigl(G(n; m)\bigr) = k$, then $m \geqslant \binom{k}{2}$ and if $m = \binom{k}{2}$, then $n = k$, i.e. we have the complete graph of k vertices. Determine the smallest integer $f(l, k)$ for which there exists a graph G having $f(l, k)$ edges and satisfying $K(G) \leqslant l$, $H(G) = k$. As we just stated, $f(k, k) = \binom{k}{2}$ and Dirac showed that $f(k-1, k) = \binom{k+2}{2} - 5$ (see [3] and [4]). It seems to be very difficult to determine $f(2, k)$. The graph constructed in [6] shows that $f(2, k) < c_7 k^3 (\log k)^3$ and it is easy to see that $f(2, k) > c_8 k^3$. Perhaps

$$\lim_{k\to\infty} \frac{f(2, k)}{k^3} = C < \infty$$

exists **(P 575)**.

Denote by $g(n; l)$ the smallest integer for which there is a $G\big(n; g(n; l)\big)$ satisfying $I\Big(G\big(n; g(n; l)\big)\Big) = l$. Turán [9] determined $g(n; l)$ for every n and l. Let $g(n; k, l)$ be the smallest integer for which there is a $G\big(n; g(n; k, l)\big)$ satisfying

$$I\Big(G\big(n; g(n; k, l)\big)\Big) = l, \qquad K\Big(G\big(n; g(n; k, l)\big)\Big) = k.$$

By (8) we must have $\binom{k+l-2}{k-1} \geqslant n$. I have not succeeded in determining $g(n; k, l)$ **(P 576)**.

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