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PROBLEMS AND RESULTS ON THE CONVERGENCE AND DIVERGENCE PROPERTIES OF THE LAGRANGE INTERPOLATION POLYNOMIALS AND SOME EXTREMAL PROBLEMS

by

P. ERDŐS

Budapest

In this note I will mainly discuss the joint work of Turán and myself and some of my own results and I do not claim to give a survey of the whole subject.

Let $-1 \leq x_1 < \ldots < x_n \leq 1$ be n points in $(-1, +1)$. Denote by $l_k(x)$ the fundamental functions of Lagrange interpolation, we have

$$l_k(x) = \frac{\omega(x)}{\omega'(x_k)(x - x_k)}, \quad \omega(x) = \prod_{k=1}^{n} (x - x_k).$$

It is well known that the sum $\sum_{k=1}^{n} |l_k(x)|$ plays a fundamental role in the study of the convergence and divergence properties of the Lagrange interpolation polynomials. I proved [3], [4] sharpening previous results of Faber, Bernstein and others that for every $\varepsilon > 0$ there is an $\eta > 0$ so that for $n > n_0(\varepsilon, \eta)$ the measure of the set in x for which

$$(1) \qquad \sum_{k=1}^{n} |l_k(x)| < \eta \log n$$

is less than ε, further

$$\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)| > \frac{2}{\pi} \log n - c_1. \tag{2}$$

Both (1) and (2) are in some sense best possible. It is well known that if the x_k are the roots of the n-th Tchebicheff polynomial $T_n(x)$ then

$$\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)| < \frac{2}{\pi} \log n + c_2.$$

An interesting unsolved problem asks the determination of the set $-1 \leqq x_1 < \ldots < x_n \leqq 1$ for which $\max\limits_{-1 \leqq x \leqq 1} \sum\limits_{k=1}^{n} |l_k(x)|$ is minimal. It seems likely that this set of points is characterized by the property that the values of the $n+1$ local maxima of $\sum\limits_{k=1}^{n} |l_k(x)|$ are all equal $(-1 = x_0,$ $1 = x_{n+1})$. As far as I know this conjecture is still unproved. Perhaps this conjecture will be easier to prove in case the x_i are on the unit circle and we want to minimise $\max\limits_{|z|=1} \sum\limits_{k=1}^{n} |l_k(z)|$. It seems certain that the x_i must be the n-th roots of unity.

Another related problem is to determine the set $-1 \leqq x_1 < \ldots <$ $< x_n \leqq 1$ for which

$$\min_{0 \leqq l \leqq n} \max_{x_l < x < x_{l+1}} \sum_{k=1}^{n} |l_k(x)| \tag{3}$$

is maximal. It seems likely that the solution of the two problems coincide and again the $n+1$ maxima in (3) have to be equal. I cannot prove here the analogue of (2), I can only show that [5]

$$\min_{0 \leqq l \leqq n} \max_{x_l < x < x_{l+1}} \sum_{k=1}^{n} |l_k(x)| < \sqrt{n}$$

it seems certain that $\sqrt{n}$ can be replaced by $c_3 \log n$.

(1) easily implies

$$\int_{-1}^{+1} \sum_{k=1}^{n} |l_k(x)|\, dx > c_4 \log n. \tag{4}$$

TURÁN asked the question: for which set $-1 \leqq x_1 < \ldots < x_n \leqq 1$ is the integral (4) minimal? This question does not seem to be easy, but it seems certain that asymptotically the minimum is assumed for the roots of $T_n(x)$ the n-th Tchebicheff polynomial.

I thought that the minimum of

$$\int_{-1}^{+1} \left(\sum_{k=1}^{n} l_k^2(x) \right) dx \tag{5}$$

is assumed if the x_i are the roots of the integral of the Legendre polynomial $P_{n-1}(x)$. (FEJÉR proved that $\max\limits_{-1 \leqq x \leqq 1} \sum\limits_{k=1}^{n} l_k^2(x) = 1$ holds if and only if the x_i are the roots of the integral of $P_{n-1}(x)$ [13]). SZABADOS [22] proved that this is false for every $n > 3$. It can be shown that the integral in (4) is certainly greater than $2 - \frac{c \log n}{n}$, but this result is far from being best possible,

G. GRÜNWALD [15] and J. MARCINKIEWICZ [18] proved that there exists a continuous function $f(x)$ so that the sequence $\mathcal{L}_n[f(x)]$ of Lagrange interpolation polynomials taken at the roots of $T_n(x)$ diverge everywhere. Let $x_i^{(n)}$, $-1 \leqq x_1^{(n)} < \ldots < x_n^{(n)} \leqq 1$, $n = 1, 2, \ldots$ be any point group. It easily follows from (1) that for almost all x_0 there is a continuous function $f(x)$ so that $\mathcal{L}_n[f(x_0)]$ diverges. But in fact I can prove more. I can show that there is a continuous function $f(x)$ so that $\mathcal{L}_n[f(x)]$ diverges for almost all values of x. The proof is difficult and has not yet been published. The following remark might be of some interest: FABER [12] was the first to prove that for every $-1 \leqq x_1 < \ldots < x_n \leqq 1$ there is a polynomial $P_{n-1}(x)$ of degree at most $n-1$, for which

$$|P_{n-1}(x_i)| \leqq 1,\ i = 1, \ldots, n, \text{ but } \max_{-1 \leqq x \leqq 1} |P_{n-1}(x)| > c \log n.$$

I believe I can prove that for every A there is an $\varepsilon > 0$ so that if $n > n_0(A, \varepsilon)$, there is a polynominal $P_{n-1}(x)$ of degree at most $n-1$ satisfying

$$|P_{n-1}(x_i)| \leqq 1, \qquad i = 1, \ldots, n$$

and the measure of the set in $-1 < x < 1$ for which $|P_{n-1}(x)| > A$ is greater than ε. It is easy to see that in some sense this result is best possible e.g. if the x_i are the roots of the n-th Legendre polynomial then if $A \to \infty$, ε must tend to 0 (since as is well known $\int_{-1}^{+1} P_{n-1}^2(x) \leq 2$). The largest permitted choice of $\varepsilon = \varepsilon(A)$ valid for any set $-1 \leqq x_1 < < \ldots < x_n \leqq 1$ will be very hard to determine.

Let $\varepsilon > 0$, $-1 \leqq a < b \leqq 1$. Then if $n > n_0(\varepsilon, a, b)$ and $-1 \leqq x_1 < \ldots < x_n \leqq 1$,

$$\max_{a < x < b} \left| \sum_{k=1}^{n} l_k(x) \right| > \left(\frac{2}{\pi} - \varepsilon \right) \log n. \tag{6}$$

The proof of (6) is complicated and unpublished (it sharpens a previous result of S. Bernstein). (6) immediately implies that for any point group $x_i^{(n)}$ there is an x_0, $-1 < x_0 < 1$, for which

$$\limsup_{n=\infty} \frac{1}{\log n} \sum_{k=1}^{n} |l_k^{(n)}(x_0)| \geqq \frac{2}{\pi}, \tag{7}$$

and in fact the set of these x_0's are everywhere dense. Perhaps (7) holds for almost all x_0.

It would be of interest to know if for every point group there is an x_0 in $(-1, +1)$ for which

$$\sum_{k=1}^{n} |l_k^{(n)}(x_0)| > \frac{2 \log n}{\pi} - c$$

holds for infinitely many values of n. This question does not seem to be easy.

Now we discuss some convergence and divergence phenomena of the interpolation of polynomials.

The following result of mine is of interest since it shows in view of a well known result of Fejér a suprising contrast with the behaviour of the Fourier series. Let $x_0 = \cos \frac{p}{q}$, $p \equiv q \equiv 1 \pmod 2$. Then there is a continuous function $f(x)$ so that $|\mathcal{L}_n[f(x_0)]| \to \infty$ as $n \to \infty$, where $\mathcal{L}_n[f(x)]$ is the Lagrange interpolation polynomial of $f(x)$ taken at the roots of $T_n(x)$, [6], [7]. In fact in [7] it is stated without proof that if A is any closed set then there is a continuous function $f_A(x)$ so that the set of limit points of $\mathcal{L}_n[f_A(x_0)]$ is precisely the set A. The set A can be chosen to be $+\infty$, i.e. there is an $f(x)$ for which $\mathcal{L}_n[f(x_0)] \to +\infty$.

A previous result of MARCINKIEWICZ [19] states that if the fundamental points of the interpolation are the roots of $U_n(x) = T'_{n+1}(x)$, then for every x_0 there is a sequence n_k for which $\mathcal{L}_{n_k}[f(x_0)] \to f(x_0)$. TURÁN and I [6] showed that the same result hods for the roots of $T_n(x)$, if

$$x_0 \neq \cos \frac{p}{q},\ p \equiv q \equiv 1 \pmod 2.$$

All these results show that from the point view of convergence the Lagrange interpolation polynomials behave rather badly. We just mention here a result of Turán and myself which points in the opposite direction: Let $p(x) > c > 0$ be intergrable in $(-1, +1)$ and let $P_n(x)$ be the sequence of polynomials orthogonal with respect to $p(x)$ in $(-1, +1)$. Let $f(x)$ be bounded and Riemann integrable and $\mathcal{L}_n[f(x)]$ the Lagrange interpolation polynomial of $f(x)$ taken at the roots of $P_n(x)$. Then [10],

$$\int_{-1}^{+1} \{f(x) - \mathcal{L}_n[f(x)]\}^2 \, dx \to 0.$$

We have not been able to find usable necessary and sufficient conditions for the point group in order that $\int_{-1}^{+1} \{f(x) - \mathcal{L}_n[f(x)]\}^2 \, dx$ should converge to 0. This fact should be compared with the well known necessary and sufficient conditions of HAHN and PÓLYA for the convergence of $\mathcal{L}_n[f(x_0)]$ and $\int_{-1}^{+1} \mathcal{L}_n[f(x)] \, dx$, [17], [21].

MARCINKIEWICZ [20] and independently FELDHEIM and I [11] proved that if the fundamental points are the roots of $T_n(x)$ then for every Riemann integrable and bounded $f(x)$ and $r > 0$,

$$\int_{-1}^{+1} \{f(x) - \mathcal{L}_n[(f(x)]\}^{2r} \, dx \to 0. \tag{8}$$

As far as I know (8) has not been proved for any other point group than the roots of $T_n(x)$. Probabily (8) holds for the roots of all Jacobi polynomials both parameters of which are in $\left(-1, -\frac{1}{2}\right)$, Feldheim showed that (8) fails for $r = 2$ for the roots of $U_n(x)$.

A well known result of FEJÉR [14] states that the Hermite interpolation parabolas converge uniformly to the continuous $f(x)$ in $(-1, +1)$ if the point group is the roots of $T_n(x)$. Fejér in fact proved this for many other point groups, for a more general result see G. GRÜNWALD [16]. S. BERNSTEIN [1] proved the following result: Let $f(x)$, $-1 \leq x \leq 1$ be any continuous function. Then to every $c > 0$ there is a sequence of polynomials $\psi_{n-1}(x)$ of degree $\leq n-1$ for which $\psi_{n-1}(x_i^{(n)}) = f(x_i^{(n)})$ for at least $n(1-c)$ roots of $T_n(x)$ and $\psi_{n-1}(x) \to f(x)$ uniformly in $(-1, +1)$.

In this direction I proved [8] the following very much more general results which in some sense give the final answer to these questions. To formulate our results we first have to introduce some notations. Put $\cos \vartheta_i^{(n)} = x_i^{(n)}$ and denote by $N_n(a, b)$, $0 \leq a < b \leq \pi$ the number of the $\vartheta_i^{(n)}$ in (a, b).

THEOREM 1. *Let* $x_i^{(n)}, i = 1, \ldots, n;\ n = 1, \ldots$ *be a point group. The necessary and sufficient condition that to every continuous function* $f(x)$ *and to every* $c > 0$ *there exists a sequence of polynomials* $\varphi_m(x)$ *of degree* $m < n(1+c)$, *such that* $\varphi_m(x_i^{(n)}) = f(x_i^{(n)})$, $i = 1, \ldots, n$ *and* $\varphi_m(x) \to f(x)$ *uniformly in* $(-1, +1)$ *is that if* $n(b_n - a_n) \to \infty$, $0 \leq a_n < b_n \leq \pi$

$$\limsup_{n=\infty} \frac{N_n(a_n, b_n)}{n(b_n - a_n)} \leq \frac{1}{\pi} \tag{9}$$

and

$$\liminf_{n=\infty} n(\vartheta_{i+1}^{(n)} - \vartheta_i^{(n)}) > 0, \ i = 1, \ldots, n. \tag{10}$$

Condition (1) states that the number of $\vartheta_i^{(n)}$ in (a_n, b_n) for „large" $b_n - a_n$ cannot be much larger than the number of roots of $\cos nx = 0$.

Theorem 1 is related to, but does not generalise the theorem of Bernstein. I can prove the following result which is a direct generalisation of Bernstein's theorem:

THEOREM 2. *The necessary and sufficient condition that to every continuous* $f(x)$ *and to every* $c > 0$ *there exists a sequence of polynomials* $\psi_{n-1}(x)$ *of degree* $\leq n-1$ *such that* $\psi_{n-1}(x) \to f(x)$ *uniformly in* $(-1, +1)$ *and such that* $\psi_{n-1}(x_i^{(n)}) = f(x_i^{(n)})$ *holds for at least* $n(1-c)$ *values of* i *is that for every* $\varepsilon > 0$

$$\Sigma' N_n(a_k, b_k) = o(n), \tag{11}$$

where in Σ' *the summation is extended over an arbitrary set of disjoint „long" intervals* (i. e. $n(b_k - a_k) \to \infty$) *satisfying*

$$N(a_k, b_k) > \frac{n(b_k - a_k)}{\pi}(1 + \varepsilon) \tag{12}$$

and that (10) *is violated for at most* $o(n)$ *values of* i.

Roughly speaking our conditions require that (9) and (10) should be nearly always satisfied.

Theorem 2 is not even stated in [8] but can be proved by the methods of [8]. In [8] I also give a necessary and sufficient condition for the point group in order that there should exist a sequence of polynomials $\varphi_m(x) \to f(x)$ of degree $m < dn$ satisfying $\varphi_m(x_i^{(n)}) = f(x_i^{(n)})$ for $i = 1, \ldots, n$.

Our results with TURÁN [9] imply that if

$$(13) \qquad |l_k(x)| < c_5, \ -1 \leq x \leq 1, \quad k = 1, \ldots, n, \ n = 1, \ldots$$

is assumed then (9) and (10) are satisfied, thus our theorems apply if (13) is satisfied, and in fact Theorem 1 is proved under the assumption (13) in [8] in a very simple way, the proof of Theorems 1 and Theorem 2 in their full generality is rather complicated.

Finally I would like to state a result of BERNSTEIN [2] and some of its generalisations:

Let $m > n(1+c)$ the x_i, $1 \leq i \leq m$ are the roots of $T_m(x)$, $P_n(x)$ is a polynomial of degree n satisfying $|P_n(x_i)| \leq 1$, $i = 1, \ldots, m$. Then

$$\max_{-1 \leq x \leq 1} |P_n(x)| < A = A(c).$$

ZYGMUND [23] proved the same result if the x_i are the roots of the n-th Legendre polynomial.

In a recent paper [9] I proved a comprehensive generalisation of these results. First we have to introduce some notations. As before put $\cos x_i^{(m)} = \vartheta_i^{(m)}$ and let

$$\alpha \leq \vartheta_i^{(m)} < \vartheta_{i+1}^{(m)} < \ldots < \vartheta_j^{(m)} \leq \beta,$$

be the $\vartheta^{(m)}$'s in (α, β). Clearly $N_m(\alpha, \beta) = j - i + 1$. For each $\eta > 0$ we now define a subsequence of these $\vartheta^{(m)}$'s. Put $\vartheta_{i_1}^{(m)} = \vartheta_i^{(m)}$ and assume that $\vartheta_{i_1}^{(m)} < \ldots < \vartheta_{i_r}^{(m)}$ has already been defined, then $\vartheta_{i_{r+1}}^{(m)}$ is the least $\vartheta_s^{(m)} \geq \vartheta_{i_r}^{(m)} + \frac{\eta}{m}$. Thus we obtain $\vartheta_{i_1}^{(m)} < \ldots < \vartheta_{i_l}^{(m)}$, $\vartheta_{i_l}^{(m)} > \vartheta_j^{(m)} - \frac{\eta}{m}$. Put $N_m^{(\eta)}(\alpha, \beta) = l$.

THEOREM 3. *Let* $-1 \leq x_1^{(m)} < \ldots < x_m^{(m)} \leq 1$, $m = 1, 2, \ldots$. *Let* $P_n(x)$ *be a polynomial of degree* n *satisfying*

$$(14) \qquad |P_n(x_i^{(m)})| \leq 1, \ i = 1, \ldots, m, \ m > n(1+c).$$

The necessary and sufficient condition that (14) *should imply for every* $c > 0$

$$(15) \qquad \max_{-1 \leq x \leq 1} |P_n(x)| < A(c),$$

is that there should be an $\eta > 0$, *independent of* m, *so that for every* $\alpha_m < \beta_m$ *satisfying* $m(\beta_m - \alpha_m) \to \infty$,

$$N_m^{(\eta)}(\alpha_m, \beta_m) \geqq (1 + o(1)) \frac{m}{\pi} (\beta - \alpha). \tag{16}$$

In other words every interval large compared to $\frac{1}{m}$ contains asymptotically at least as many points $\vartheta_i^{(m)}$, no two of which are too close, as the roots of $\cos mx$.

Theorems 1 and 2 follow from the following results on polynomials.

THEOREM 1′. *The necessary and sufficient condition that there should exist to every* $c > 0$ *an* $A(c)$ *so that to every* $|y_i^{(n)}| \leq 1$, $i = 1, \ldots, n$, $n = 1, \ldots$, *there should exist a polynomial* $P_m(x)$ of degree $m < (1 + c)n$, *satisfying*

$$P_m(x_i^{(n)}) = y_i^{(n)}, \ \ i = 1, \ldots, n; \max_{-1 \leqq x \leqq 1} |P_m(x)| < A(c)$$

is that (9) *and* (10) *should hold.*

THEOREM 2′. *The necessary and sufficient condition that there should exist to every* $c > 0$ *an* $A(c)$ *so that for every* $|y_i^{(n)}| \leqq 1$, $i = 1, \ldots, n$, *there should exist a polynomial* $P_{n-1}(x)$ *of degree* $\leqq n - 1$ *satisfying* $P_{n-1}(x_i^{(n)}) = y_i^{(n)}$ *for at least* $n(1 - c)$ *values of* i *and* $\max\limits_{-1 \leqq x \leqq 1} |P_{n-1}(x)| < A(c)$ *is that* (11) *should hold and* (10) *should be violated for at most* $o(n)$ *values of* i.

Theorem 1′ is stated in [9]. In [9] I state without proof the following.

THEOREM 4. *To every* A *however large there is an* $\varepsilon > 0$ *so that if* $n > n_0(A, \varepsilon)$, $m = [(1 + \varepsilon)n]$, *then for every* $-1 \leqq x_1 < \ldots < x_m \leqq 1$ *there is a polynomial of degree* n, $P_n(x)$, *satisfying*

$$|P_n(x_i)| \leqq 1, \ i = 1, \ldots, m \text{ and } \max_{-1 \leqq x \leqq 1} |P_n(x)| > A.$$

Theorem 4 clearly sharpens the well known result of FABER [12] (in the theorem of Faber $m = n + 1$). Theorem 4 shows that in Theorem 3 $m > n(1 + c)$ can never be weakened to $m > n(1 + o(1))$.

Probably the following result also holds:

To every A however large there is an $\varepsilon > 0$ so that if $n > n_0(A, \varepsilon)$ then for every $-1 \leqq x_1 < \ldots < x_n \leqq 1$ there is a set $y_1, \ldots, y_n$, $|y_i| \leqq 1$, $i = 1, \ldots, n$ so that every polynomial $P_m(x)$ of degree $m < (1 + \varepsilon)n$ for which $P_m(x_i) = y_i$ holds for at least $n(1 - \varepsilon)$ values of i satisfies $\max\limits_{-1 \leqq x \leqq 1} |P_m(x)| > A$.

This result if true clearly contains Theorem 4. I have not even proved it if $m = n$.

REFERENCES

[1] S. Bernstein, *Sur une modification de la formule d'interpolation de Lagrange*. Comm. Soc. Math. Kharkow, (4), **5**, 49–57 (1932).

[2] S. Bernstein, *Sur une classe de formules d'interpolation*. Bull. Acad. Sci. U. S. S. R. (**7**), **4**, 1151–1161 (1931).

[3], [4] P. Erdős, *Problems and on the theory of interpolation*, (I) and (II). Acta Math. Acad. Sci. Hung. **9**, 381–388 (1958), and ibid, **12**, 235–244 (1961).

[5] — *Some remarks on polynomials*. Bull. Amer. Math. Soc., **53**, 1169–1176, see p. 1171 (1947).

[6], [7] — *On divergence properties of the Lagrange interpolation parabolas*. Annals of Math., **42**, 309–315 (1941) and *Corrections to two of my papers*. ibid. **44**, 647–51 (1943).

[8] — *On Some convergence properties of the interpolation polynomials*. Annals of Math., **44**, 330–337 (1943).

[9] — *On the boundedness and unboundedness of polynomials*. Journal d'Analyse, **18**.

[10] P. Erdös and P. Turán, *On interpolation* (I), *Quadrature and mean convergence in the Lagrange interpolation*. Annals of Math. **38**, 142–155 (1937).

[11] P. Erdös and E. Feldheim, *Sur le mode de convergence pour l'interpolation de Lagrange*. Comptes Rendus, **203**, 913–915 (1936).

[12] G. Faber, *Über die interpolatorische Darstellung stetiger Funktionen*. Jahresbericht der Deutschen Math. Ver., **23**, 190–210 (1914).

[13] L. Fejér, Annali della R. Scuola Normale Sup. di Pisa, II, 1, 263–276 (1932).

[14] — Math. Zeitschrift, **32**, 426–457 (1930).

[15] G. Grünwald, *Über die Divergenzerscheinungen* etc., Annals of Math., **37**, 908–918 (1936).

[16] — *On the theory of interpolation*. Acta Math, **75** (1943).

[17] H. Hahn, *Über das Interpolationsproblem*. Math. Zeitschrift **1**, 115–142 (1918).

[18] J. Marcinkiewicz, Sur la divergence des polynômes d'interpolation, Acta Sci. Math. Szeged, **8**, 131–135 (1937).

[19] — Acta Sci. Math. Szeged, **8**, 127–130 (1937).

[20] — *Sur l'interpolation* (I). Studia Math., **6**, 1–17(1936), See also „Collected papers", Warszawa, Panst. Wyd. Nauk (1961).

[21] G. Pólya, *Über die Konvergenz von Quadraturverfahren*. Math. Zeitschrift **37**, 264–287 (1933).

[22] I. Szabados, *On a problem of Erdős*. Acta Math. Ac. Sci. Hung., *17*, 155–157, (1966).

[23] A. Zygmund, *A property of the zeros of Legendre polynomials*. Trans. Amer. Math. Soc., **54**, 39–56 (1943).

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