Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 18 (3—4), 1967, pp. 309—320.

ON SOME PROBLEMS OF A STATISTICAL GROUP-THEORY. III

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1. Let S_n be the symmetric group of n letters, P a generic element of it and $\mathbf{O}(P)$ its order (as group-element). As E. LANDAU proved* denoting by $G(n)$ the maximum of $\mathbf{O}(P)$ for $P \in S_n$ the relation

$$\lim_{n\to\infty} \frac{\log G(n)}{\sqrt{n \log n}} = 1$$

holds. In our first paper in this series** we proved that for almost all P's (i.e. with exception of $o(n!)$ P's) the much stronger inequality

$$|\log \mathbf{O}(P) - \tfrac{1}{2} \log^2 n| < \omega(n) \log^{\frac{3}{2}} n$$

holds if only $\omega(n)$ tends to infinity with n arbitrarily slowly and we expressed the conjecture that $\log \mathbf{O}(P)$ shows a Gaussian distribution. In this paper we are going to prove essentially this conjecture. This proof rests heavily on the inequality (14. 4) of P. I, which we are going to expose detailed in **2**; otherwise this paper can be read independently of P. I (and also from P. II).

More exactly, we are going to prove the following

THEOREM. *Denoting by $K(n, x)$ the number of P's in S_n satisfying the inequality*

$$\log \mathbf{O}(P) \leqq \tfrac{1}{2} \log^2 n + x \log^{\frac{3}{2}} n$$

the relation

$$\lim_{n\to\infty} \frac{K(n, x)}{n!} = \sqrt{\frac{3}{2\pi}} \int_{-\infty}^{x} e^{-\frac{3}{2}\lambda^2} \, d\lambda \tag{1. 1}$$

*holds, uniformly for $-x_0 \leqq x \leqq x_0$, x_0 being an arbitrarily large positive number.****

* See his *Handbuch der Lehre von der Verteilung der Primzahlen*, (1909), Bd. I. p. 222.

** *Zeitschr. f. Wahrscheinlichkeitstheorie und verw. Gebiete*, **4** (1965), pp. 175—186. Quoted later as P. I.

*** As A. RÉNYI remarked the relation (1. 1) can be written in the more elegant form

$$\text{Prob}\left(\frac{\log \mathbf{O}(P) - \frac{1}{2} \log^2 n}{\frac{1}{\sqrt{3}} \log^{\frac{3}{2}} n} < y \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{v^2}{2}} \, dv.$$

Our proof is a direct one and rather long; but a *first* proof can be as long as it wants to be. It would be however of interest to deduce it from the general principles of probability theory. Obviously our proofs could be modified so that they could replace (1. 1) by an *explicit* inequality, even with a main-term plus an explicit error-term. As to this we shall not enter into details.

2. The different cycle-lengths in the canonical cycle-decomposition of P will be denoted by

$$(1\leqq)n_1<n_2<\dots<n_k \qquad k=k(P) \tag{2. 1}$$

and their multiplicity by $m_1, m_2, \dots, m_k$ ($\geqq 1$), respectively, so that

$$m_1 n_1+\dots+m_k n_k=n. \tag{2. 2}$$

Then the crucial inequality (14. 4) from P. I asserts that for all but $o(n!)$ P's the inequality

$$\exp\left(-3\log n(\log\log n)^4\right)\leqq\frac{\mathbf{O}(P)}{n_1 n_2\dots n_k}\leqq 1 \tag{2. 3}$$

holds.

Also here we shall use the fact, known to Cauchy, that fixing the n_ν and m_ν numbers with (2. 2) as above, the number of P's in S_n having m_ν cycles of length n_ν ($\nu=1, 2, \dots, k$) in the canonical cycle-decomposition is

$$\frac{n!}{m_1!\,m_2!\,\dots\,m_k!\,n_1^{m_1}n_2^{m_2}\dots n_k^{m_k}} \tag{2. 4}$$

and also an easy consequence of it, namely that

$$\sum_{k=1}^{\infty}\sum_{m_\nu}\sum_{n_\nu}\frac{1}{m_1!\,m_2!\,\dots\,m_k!\,n_1^{m_1}\dots n_k^{m_k}}=1, \tag{2. 5}$$

where the summation is extended to all systems satisfying (2. 1)—(2. 2).

In what follows c will denote positive constant, not necessarily the same in different occurences, which may depend at most on t_0 (in (4. 6)). If for $|z|<1$

$$f(z)=\sum_{\nu=0}^{\infty}a_\nu z^\nu,$$

$$g(z)=\sum_{\nu=0}^{\infty}b_\nu z^\nu, \qquad b_\nu\geqq 0$$

we shall use the notation

$$f(z)\ll g(z) \tag{2. 6}$$

if and only if

$$|a_n|\leqq b_n \qquad (n=0, 1, 2, \dots). \tag{2. 7}$$

Some positive numerical constants will be denoted by $d_1, d_2, \dots$. Sometimes we use the O-sign which refers to $n\to\infty$, depending perhaps on t_0.

3. We shall need two simple lemmata.

LEMMA I. *We have*

$$\sum_{m=2}^{\infty} \frac{\log m}{m} z^m - \frac{1}{2} \log^2 \frac{1}{1-z} \ll c \log \frac{1}{1-z}.$$

For the simple proof we remark only that $m \geqq 2$

$$\text{coeffs of } z^m \text{ in } \log^2 \frac{1}{1-z} = \sum_{\nu=1}^{m-1} \frac{1}{\nu(m-\nu)} = \frac{2}{m} \sum_{\nu=1}^{m-1} \frac{1}{\nu} = 2\frac{\log m}{m} + O\left(\frac{1}{m}\right).$$

LEMMA II. *We have*

$$\sum_{m=2}^{\infty} \frac{\log^2 m}{m} z^m - \frac{1}{3} \log^3 \frac{1}{1-z} \ll c \sum_{m=2}^{\infty} \frac{\log m \log\log m}{m} z^m.$$

Among the several possibilities to prove it, possibly the shortest one is based on complex integration (and part of which can be used later too). Obviously

$$(3.1) \qquad J_m \overset{\text{def}}{=} \text{coeffs of } z^m \text{ in } \log^3 \frac{1}{1-z} = \frac{1}{2\pi i} \int_{(l)} z^{-m-1} \log^3 \frac{1}{1-z}\, dz$$

where l encircles the origin in $|z| < 1$. Let L be the loop along the segment $1 \leqq z < \infty$. Trivial estimations lead to

$$(3.2) \qquad J_m = \frac{1}{2\pi i} \int_{(L)} z^{-m-1} \log^3 \frac{1}{1-z}\, dz.$$

On the upper part of the cut

$$(3.3) \qquad \log \frac{1}{1-z} \to \log \frac{1}{r-1} + i\pi,$$

on the lower one

$$(3.4) \qquad \log \frac{1}{1-z} \to \log \frac{1}{r-1} - i\pi$$

with the positive value of the logarithm. Routine-estimation gives then

$$(3.5) \quad J_m = |J_m| = \int_1^{\infty} \frac{3\log^2 \frac{1}{r-1} - \pi^2}{r^{m+1}}\, dr = -\frac{\pi^2}{m} + 3\int_0^{\infty} (1+x)^{-m-1} \log^2 \frac{1}{x}\, dx.$$

As to the remaining integral this is

$$> \int_0^{10m^{-1}\log m} (1+x)^{-m-1} \log^2 \frac{1}{x}\, dx > \frac{1}{m} \log^2 \left(\frac{m}{10 \log m}\right) \cdot$$

$$\cdot \left\{1 - \left(1 + \frac{10 \log m}{m}\right)^{-m}\right\} > \frac{\log^2 m}{m} - c\,\frac{\log m \log\log m}{m}$$

and — apart from $O\left(\frac{\log m \log\log m}{m}\right)$ — the same upper bound can analogously be obtained. These prove the lemma.

4. Now we can turn to the proof of our theorem.

First step. Let x be real and

$$F_n(x) \overset{\text{def}}{=} \sum_{k=1}^{\infty} \sum_{m_\nu}{}' \sum_{n_\nu}{}' \frac{1}{m_1!\, m_2! \dots m_k!\, n_1^{m_1} n_2^{m_2} \dots n_k^{m_k}} \tag{4.1}$$

where the summation is extended to all (m_ν, n_ν)-systems with

$$(1 \leqq) n_1 < n_2 < \dots < n_k, \tag{4.2}$$

$$m_\nu \geqq 1, \tag{4.3}$$

$$\sum_{\nu=1}^{k} m_\nu n_\nu = n, \tag{4.4}$$

$$n_1 n_2 \dots n_k \leqq \exp\{\tfrac{1}{2}\log^2 n + x \log^{\frac{3}{2}} n\}. \tag{4.5}$$

This is an increasing function with $F_n(-\infty) = 0$ and owing to (2. 5) also $F_n(+\infty) = 1$. Let us consider the characteristic function

$$\varphi_n(t) = \int_{-\infty}^{\infty} e^{itx}\, dF_n(x) \qquad (-t_0 \leqq t \leqq t_0). \tag{4.6}$$

This gives for all real t's at once

$$|\varphi_n(t)| \leqq 1. \tag{4.7}$$

To get an alternative form of $\varphi_n(t)$ we use the form (4. 1) of $F_n(x)$. This gives

$$\varphi_n(t) = \sum_{k=1}^{\infty} \sum_{m_\nu} \sum_{n_\nu} \frac{1}{m_1!\, m_2! \dots m_k!} \cdot \frac{1}{n_1^{m_1} n_2^{m_2} \dots n_k^{m_k}} \cdot \exp\left\{ it \frac{\sum_{\nu=1}^{k} \log n_\nu - \frac{1}{2}\log^2 n}{\log^{\frac{3}{2}} n} \right\},$$

where the summation is extended to all (m_ν, n_ν)-systems with (4. 2)—(4. 3)—(4. 4). Hence putting

$$\frac{t}{\log^{\frac{3}{2}} n} = \tau, \tag{4.8}$$

$$\varphi_n^*(\tau) \overset{\text{def}}{=} \sum_{k=1}^{\infty} \sum_{m_\nu}{}' \sum_{n_\nu} \frac{(n_1 n_2 \dots n_k)^{i\tau}}{m_1! \dots m_k!\, n_1^{m_1} n_2^{m_2} \dots n_k^{m_k}}, \tag{4.9}$$

we have

$$\varphi_n(t) = \exp\left(-\frac{it}{2}\sqrt{\log n}\right) \varphi_n^*(\tau). \tag{4.10}$$

5. Next we try to find a suitable representation for $\varphi_n^*(\tau)$.

Second step. We form for $|z|<1$ the generating function

$$D(z,\tau)=1+\sum_{n=1}^{\infty}\varphi_n^*(\tau)z^n. \tag{5. 1}$$

Putting $n=m_1 n_1+\ldots+m_k n_k$ in the expression of $\varphi_n^*(\tau)$ in (4. 9) we obtain

$$D(z,\tau)=1+\sum_{k=1}^{\infty}\sum_{n_1 n_2\ldots n_k}(n_1 n_2\ldots n_k)^{i\tau}\sum_{m_1,\ldots,m_k}\frac{1}{m_1!}\left(\frac{z^{n_1}}{n_1}\right)^{m_1}\cdots\frac{1}{m_k!}\left(\frac{z^{n_k}}{n_k}\right)^{m_k}=$$

$$=1+\sum_{k=1}^{\infty}\sum_{n_1,\ldots,n_k}(n_1 n_2\ldots n_k)^{i\tau}\left(e^{\frac{z^{n_1}}{n_1}}-1\right)\left(e^{\frac{z^{n_2}}{n_2}}-1\right)\ldots\left(e^{\frac{z^{n_k}}{n_k}}-1\right)=\prod_{l=1}^{\infty}\left\{1+l^{i\tau}\left(e^{\frac{z^l}{l}}-1\right)\right\}.$$

The l^{th} factor can be written as

$$\left\{1+(l^{i\tau}-1)\left(1-e^{-\frac{z^l}{l}}\right)\right\}e^{\frac{1}{l}z^l};$$

since for $|z|<1$

$$\prod_{l=1}^{\infty}\exp\left(\frac{z^l}{l}\right)=\frac{1}{1-z},$$

we get here

$$D(z,\tau)=\frac{1}{1-z}\prod_{l=2}^{\infty}\left\{1+(l^{i\tau}-1)\left(1-e^{-\frac{1}{l}z^l}\right)\right\}.$$

Since the factors belonging to $l\geqq n+1$ obviously do not contribute to the coefficient of z^n in $D(z,\tau)$ we obtain the required representation of $\varphi_n(t)$,

$$\varphi_n(t)=\exp\left\{-\frac{it}{2}\sqrt{\log n}\right\}\cdot\text{coeffs of } z^n \text{ in }\frac{1}{1-z}\prod_{l=2}^{\infty}\left\{1+(l^{i\tau}-1)\left(1-e^{-\frac{1}{l}z^l}\right)\right\}= \tag{5. 2}$$

$$=\exp\left\{-\frac{it}{2}\sqrt{\log n}\right\}\cdot\text{coeffs of } z^n \text{ in }\frac{1}{1-z}\prod_{l=2}^{n}\left\{1+(l^{i\tau}-1)\left(1-e^{-\frac{1}{l}z^l}\right)\right\}.$$

6. Next we simplify the expression

$$D_n(z,\tau)\overset{\text{def}}{=}\sum_{l=2}^{n}\log\left\{1+(l^{i\tau}-1)\left(1-e^{-\frac{1}{l}z^l}\right)\right\}. \tag{6. 1}$$

Third step. Obviously $D_n(z,\tau)$ can be written as the sum of the following four functions

$$h_1(z)=\sum_{l=2}^{n}\left\{i\tau\frac{\log l}{l}-\frac{\tau^2}{2}\frac{\log^2 l}{l}\right\}z^l, \tag{6. 2}$$

$$h_2(z)=\sum_{l=2}^{n}\left(\frac{l^{i\tau}-1}{l}-i\tau\frac{\log l}{l}+\frac{\tau^2}{2}\frac{\log^2 l}{l}\right)z^l, \tag{6. 3}$$

$$h_3(z)=\sum_{l=2}^{n}(l^{i\tau}-1)\left(1-e^{-\frac{1}{l}z^l}-\frac{z^l}{l}\right), \tag{6. 4}$$

$$h_4(z)=\sum_{l=2}^{n}\sum_{j=2}^{\infty}\frac{(-1)^{j-1}}{j}(l^{i\tau}-1)^j\left(1-e^{-\frac{1}{l}z^l}\right)^j. \tag{6. 5}$$

Owing to (4. 8) we get

$$h_4(z) \ll \sum_{l=2}^{n} \sum_{j=2}^{\infty} \left(\frac{2t_0}{\sqrt{\log n}}\right)^j \left(e^{\frac{1}{l}z^l}-1\right)^j \ll \sum_{l=2}^{n} \sum_{j=2}^{\infty} \left(\frac{2t_0}{\sqrt{\log n}}\right)^j \left(\frac{z^l}{l-z^l}\right)^j =$$

$$= \frac{(2t_0)^2}{\log n} \sum_{l=2}^{n} \frac{z^{2l}}{l-z^l} \cdot \frac{1}{l-\left(1+\frac{2t_0}{\sqrt{\log n}}\right)z^l} \ll \frac{(2t_0)^2}{\log n} \sum_{l=2}^{\infty} \frac{z^{2l}}{l^2} \frac{1}{\left\{1-\left(1+\frac{2t_0}{\sqrt{\log n}}\right)\frac{z^l}{l}\right\}^2} =$$

$$= \frac{(2t_0)^2}{\log n} \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} (m+1) \left(\frac{1+\frac{2t_0}{\sqrt{\log n}}}{l}\right)^{m+2} \cdot z^{l(m+2)}$$

so that for $n > \exp(10^4 t_0^2)$ and $v \geqq 4$

$$\text{(6. 5)} \qquad |\text{coeffs of } z^v \text{ in } h_4(z)| \leqq \frac{(2t_0)^2}{\log n} \sum_{\substack{(m+2)l=v \\ l \geqq 2,\, m \geqq 0}} (m+1) \left(\frac{1{,}02}{l}\right)^{\frac{v}{l}} <$$

$$< \frac{(2t_0)^2}{\log n} \sum_{\substack{l|v \\ 2 \leqq l \leqq \frac{v}{2}}} \frac{v}{l} \left(\frac{1{,}02}{l}\right)^{\frac{v}{l}} \overset{\text{def}}{=} Z_1.$$

Since the possible divisors of v between $v/4$ and $v/2$ are $v/3$ and $v/2$, we get for $v > c$

$$\text{(6. 6)} \qquad Z_1 < \frac{(2t_0)^2}{\log n} \left\{ 2\left(\frac{2{,}04}{v}\right)^2 + 3\left(\frac{3{,}06}{v}\right)^3 + \right.$$

$$\left. + \sum_{2 \leqq l \leqq \frac{v}{10 \log v}} \frac{v}{l} \left(\frac{1{,}02}{l}\right)^{\frac{v}{l}} + \sum_{\frac{v}{10 \log v} < l \leqq \frac{v}{4}} \frac{v}{l} \left(\frac{1{,}02}{l}\right)^{\frac{v}{l}} \right\} < \frac{c}{v^2 \log n}.$$

This holds for $v < c$, too. As to $h_3(z)$ we have

$$h_3(z) \ll \sum_{l=2}^{n} \frac{2t_0}{\sqrt{\log n}} \left(e^{\frac{1}{l}z^l} - 1 - \frac{z^l}{l}\right) \ll \sum_{l=2}^{n} \frac{2t_0}{\sqrt{\log n}} \sum_{m=2}^{\infty} \left(\frac{z^l}{l}\right)^m$$

and hence for $v \geqq 4$

$$\text{(6. 7)} \qquad |\text{coeffs of } z^v \text{ in } h_3(z)| \leqq \frac{2t_0}{\sqrt{\log n}} \sum_{\substack{m|v \\ 2 \leqq m \leqq \frac{v}{2}}} \left(\frac{m}{v}\right)^m < \frac{c}{v^2 \sqrt{\log n}}$$

as before. As to $h_2(z)$ we have owing to (4. 8)

$$h_2(z) \ll c \sum_{l=2}^{n} \left(\frac{2t_0}{\sqrt{\log n}}\right)^3 \frac{z^l}{l}$$

i.e. for $\nu \geqq 2$

$$|\text{coeffs of } z^\nu \text{ in } h_2(z)| < \frac{c}{\nu \log^{\frac{3}{2}} n}. \tag{6.8}$$

Collecting all these, (6. 1) gives

$$D_n(z, \tau) = i\tau \sum_{l=2}^{n} \frac{\log l}{l} z^l - \frac{\tau^2}{2} \sum_{l=2}^{n} \frac{\log^2 l}{l} z^l + \sum_{\nu=2}^{\infty} a_\nu^{(1)} z^\nu \tag{6.9}$$

with

$$|a_\nu^{(1)}| < c\left(\frac{1}{\nu^2 \sqrt{\log n}} + \frac{1}{\nu \log^{\frac{3}{2}} n}\right). \tag{6.10}$$

Remarking that

$$\text{coeffs of } z^n \text{ in } \frac{1}{1-z} \exp\{D_n(z, \tau)\} =$$

$$= \text{coeffs } z^n \text{ in } \frac{1}{1-z} \exp\left\{i\tau \sum_{l=2}^{\infty} \frac{\log l}{l} z^l - \frac{\tau^2}{2} \sum_{l=2}^{\infty} \frac{\log^2 l}{l} z^l + \sum_{l=2}^{\infty} a_\nu^{(1)} z^\nu\right\}$$

the representation (5. 2) assumes the form

$$\varphi_n(t) = \exp\left\{-\frac{it}{2}\sqrt{\log n}\right\}\cdot \tag{6.11}$$

$$\cdot \text{coeffs of } z^n \text{ in } \frac{1}{1-z} \exp\left\{i\tau \sum_{l=2}^{\infty} \frac{\log l}{l} z^l - \frac{\tau^2}{2} \sum_{l=2}^{\infty} \frac{\log^2 l}{l} z^l\right\} \exp\left\{\sum_{\nu=2}^{\infty} a_\nu^{(1)} z^\nu\right\}.$$

7. Next we want to have in the „essential'' factor of (6. 11) elementary functions only. This is performed by the

Fourth step. Here we shall use Lemmata I and II. Using (4. 8) this gives at once the modified representation of

$$\varphi_n(t) = \exp\left\{-\frac{it}{2}\sqrt{\log n}\right\}\cdot \tag{7.1}$$

$$\cdot \text{coeffs of } z^n \text{ in } \frac{1}{1-z} \exp\left\{\frac{it}{2\log^{\frac{3}{2}} n} \log^2 \frac{1}{1-z} - \frac{t^2}{6\log^3 n} \log^3 \frac{1}{1-z}\right\} \exp\left\{\sum_{\nu=2}^{\infty} a_\nu^{(2)} z^\nu\right\}$$

where again for $\nu \geqq 2$

$$|a_\nu^{(2)}| \leqq c\left(\frac{1}{\nu^2\sqrt{\log n}} + \frac{1}{\nu \log^{\frac{3}{2}} n} + \frac{\log \nu \log\log \nu}{\nu \log^3 n}\right). \tag{7.2}$$

8. If we want to hope that the factor $\exp\left\{\sum_{\nu=2}^{\infty} a_\nu^{(2)} z^\nu\right\}$ will not matter in (7. 1) we have to know that putting

$$\exp\left\{\sum_{\nu=2}^{\infty} a_\nu^{(2)} z^\nu\right\} \stackrel{\text{def}}{=} 1 + \sum_{\nu=2}^{\infty} a_\nu^{(3)} z^\nu \tag{8.1}$$

the $a_\nu^{(3)}$-coefficients are sufficiently small. This is done by the

Fifth step. By (7. 2) we have — meaning the majoration only for $\nu \leqq n$ —

$$(8.2) \qquad 1+\sum_{\nu=2}^{\infty} a_\nu^{(3)} z^\nu \ll \exp c\left\{\log^{-\frac{3}{2}} n \log\frac{1}{1-z} + \log^{-\frac{1}{2}} n \sum_{\mu=1}^{\infty} \frac{z^{\mu+1}}{\mu(\mu+1)}\right\} =$$

$$= \left(\frac{1}{1-z}\right)^{c\log^{-\frac{3}{2}} n} \cdot \exp\{c\log^{-\frac{1}{2}} n((1-z)\log(1-z)+z)\} \overset{\text{def}}{=}$$

$$= \left\{1+\sum_{\mu=1}^{\infty} a_\mu^{(4)} z^\mu\right\}\left\{1+\sum_{\nu=1}^{\infty} a_\nu^{(5)} z^\nu\right\}.$$

Obviously we have — with $c\log^{-\frac{3}{2}} n = \eta_0$ — for $\mu \leqq n$

$$(8.3) \qquad |a_\mu^{(4)}| = \frac{\eta_0}{\mu}\left(1+\frac{\eta_0}{1}\right)\left(1+\frac{\eta_0}{2}\right)\cdots\left(1+\frac{\eta_0}{\mu-1}\right) < \eta_0 \mu^{2\eta_0-1} < \frac{c}{\mu\log^{\frac{3}{2}} n}.$$

As to the $a_\nu^{(5)}$-coefficients we have — with $c\log^{-\frac{1}{2}} n = \eta_1$ — for $\nu \geqq 2$

$$a_\nu^{(5)} = \frac{1}{2\pi i}\int\limits_{|z|=1-\frac{1}{\nu}} z^{-\nu-1}\exp \eta_1\{(1-z)\log(1-z)+z\}\,dz =$$

$$= \frac{\eta_1}{2\pi i\nu}\int\limits_{|z|=1-\frac{1}{\nu}} z^{-\nu}\log\frac{1}{1-z}\exp\eta_1\{(1-z)\log(1-z)+z\}\,dz =$$

$$= \frac{\eta_1}{2\pi i\nu(\nu-1)}\int\limits_{|z|=1-\frac{1}{\nu}} z^{-\nu+1}\left\{\frac{1}{1-z}+\eta_1\log^2\frac{1}{1-z}\right\}\exp\eta_1\{(1-z)\log(1-z)+z\}\,dz$$

and hence

$$|a_\nu^{(5)}| < \frac{c}{\nu^2\sqrt{\log n}}\int\limits_{|z|=1-\frac{1}{\nu}} \frac{|dz|}{|1-z|} < \frac{c\log(\nu+1)}{\nu^2\sqrt{\log n}}.$$

From this and (8. 3) we get for $1 \leqq \nu \leqq n$

$$|a_\nu^{(3)}| = \left|a_\nu^{(4)} + a_\nu^{(5)} + \sum_{j=1}^{\nu-1} a_j^{(4)} a_{\nu-j}^{(5)}\right| <$$

$$< c\left\{\frac{1}{\nu\log^{\frac{3}{2}} n} + \frac{1}{\nu^2\sqrt{\log n}} + \frac{1}{\log^2 n}\sum_{j=1}^{\nu-1}\frac{\log(j+1)}{j^2(\nu-j)}\right\} =$$

$$= c\left\{\frac{1}{\nu\log^{\frac{3}{2}} n} + \frac{1}{\nu^2\sqrt{\log n}} + \frac{1}{\log^2 n}\left(\sum_{j\leqq\frac{\nu}{2}} + \sum_{\frac{\nu}{2}<j\leqq\nu-1}\right)\right\} < c\left(\frac{1}{\nu\log^{\frac{3}{2}} n} + \frac{1}{\nu^2\sqrt{\log n}}\right).$$

Hence the representation (7. 1)—(7. 2) takes the form

$$\varphi_n(t) = \exp\left\{-\frac{it}{2}\sqrt{\log n}\right\} \text{coeffs } z^n \text{ in } h(z)\left\{1+\sum_{\nu=1}^{\infty} a_\nu^{(3)} z^\nu\right\} \tag{8. 4}$$

with

$$h(z) \overset{\text{def}}{=} \frac{1}{1-z}\exp\left\{\frac{it}{2\log^{\frac{3}{2}} n}\log^2\frac{1}{1-z} - \frac{t^2}{6\log^3 n}\log^3\frac{1}{1-z}\right\} \tag{8. 5}$$

where for $1 \leqq \nu \leqq n$

$$|a_\nu^{(3)}| < c\left(\frac{1}{\nu\log^{\frac{3}{2}} n} + \frac{1}{\nu^2\sqrt{\log n}}\right). \tag{8. 6}$$

9. Now we turn to the study of the coefficients e_m of the Mac—Laurin series of $h(z)$ for $m \leqq n$. This will be based on the integral-representation

$$e_m = \frac{1}{2\pi i}\int_{(L_1)} z^{-m-1} h(z)\, dz \tag{9. 1}$$

where L_1 means the following path. Cutting the plane along the segment $1+\frac{1}{m} \leqq z < \infty$ it comes from $z=2$ till $z=1+\frac{1}{m}$ along the lower part of the cut, then encircles $z=1$ by the circle $|z-1| = \frac{1}{m}$ clockwise, it goes from $z=1+\frac{1}{m}$ to $z=2$ along the upper part of the cut and is finished by the arc of the circle $|z|=2$ in the cut-plane. The contribution of the circle $|z|=2$ is evidently absolutely

$$< c\cdot 2^{-m}. \tag{9. 2}$$

We shall investigate separately the contributions of the segments and the circle $|z-1| = \frac{1}{m}$, respectively.

Sixth step. Using (3. 3) and (3. 4) $h(z)$ is on the upper slit

$$= \exp\left\{\frac{it}{2}\,\frac{\left(\log\frac{1}{r-1}+i\pi\right)^2}{\log^{\frac{3}{2}} n} - \frac{t^2}{6\log^3 n}\left(\log\frac{1}{r-1}+i\pi\right)^3\right\} \tag{9. 3}$$

on the lower one

$$= \exp\left\{\frac{it}{2}\cdot\frac{\left(\log\frac{1}{r-1}-i\pi\right)^2}{\log^{\frac{3}{2}} n} - \frac{t^2}{6\log^3 n}\left(\log\frac{1}{r-1}-i\pi\right)^3\right\}. \tag{9. 4}$$

We write the expression in (9. 3) as

$$\exp\left\{\frac{it}{2}\frac{\log^2\frac{1}{r-1}}{\log^{\frac{3}{2}} n}-\frac{t^2}{6}\frac{\log^3\frac{1}{r-1}}{\log^3 n}-\frac{\pi^2 it}{2\log^{\frac{3}{2}} n}+\frac{\pi^2 t^2\log\frac{1}{r-1}}{2\log^3 n}\right\}\cdot$$

$$\cdot\exp\left\{-\frac{\pi t\log\frac{1}{r-1}}{2\log^{\frac{3}{2}} n}-\frac{i\pi t^2\log^2\frac{1}{r-1}}{2\log^3 n}+\frac{t^2\pi^3 i}{6\log^3 n}\right\}$$

and (9. 4) in the form

$$\exp\left\{\frac{it}{2}\cdot\frac{\log^2\frac{1}{r-1}}{\log^{\frac{3}{2}} n}-\frac{t^2}{6}\frac{\log^3\frac{1}{r-1}}{\log^3 n}-\frac{\pi^2 it}{2\log^{\frac{3}{2}} n}+\frac{\pi^2 t^2\log\frac{1}{r-1}}{2\log^3 n}\right\}\cdot$$

$$\cdot\exp\left\{\frac{\pi t\log\frac{1}{r-1}}{2\log^{\frac{3}{2}} n}+\frac{i\pi t^2\log^2\frac{1}{r-1}}{2\log^3 n}-\frac{t^2\pi^3 i}{6\log^3 n}\right\}.$$

The difference of these is for $1+\frac{1}{m}\leqq r\leqq 2$, $m\leqq n$ absolutely

$$<\frac{c}{\sqrt{\log n}}$$

and thus the contribution of the segment integrals is absolutely

$$(9.5)\qquad <\frac{c}{\sqrt{\log n}}\int_{1+\frac{1}{m}}^{2}\frac{dr}{(r-1)r^{m+1}}.$$

As to this last integral, we have

$$\int_{1+2\frac{\log m}{m}}^{2}\frac{dr}{r^{m+1}(r-1)}<\frac{c}{m^2}\int_{1+\frac{1}{m}}^{2}\frac{dr}{r-1}<c\frac{\log m}{m^2}$$

and further

$$\int_{1+\frac{1}{m}}^{1+2\frac{\log m}{m}}\frac{dr}{r^{m+1}(r-1)}<\int_{1+\frac{1}{m}}^{1+2\frac{\log m}{m}}\frac{dr}{r-1}<c\log\log m,$$

hence the contribution of the segments is absolutely

$$(9.6)\qquad <c\frac{\log\log n}{\sqrt{\log n}}.$$

10. *Seventh step.* We consider the integral on $|z-1| = \frac{1}{m}$. This is

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\exp\left\{\frac{it}{2\log^{\frac{3}{2}} n}(\log m + i(\pi-\varphi))^2 - \frac{t^2}{6\log^3 n}(\log m + i(\pi-\varphi))^3\right.}{\left(1+\frac{1}{m}e^{i\varphi}\right)^{m+1}}\, d\varphi$$

and hence for $m \leqq n$

$$= \frac{1}{2\pi}\exp\left\{\frac{it\log^2 m}{\log^{\frac{3}{2}} n} - \frac{t^2\log^3 m}{6\log^3 n}\right\}\cdot\int_0^{2\pi}\left(1+O\left(\frac{1}{\sqrt{\log n}}\right)\right)\cdot$$
$$\cdot\left(1+O\left(\frac{1}{m}\right)\right)\exp(-e^{i\varphi})\, d\varphi.$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi}\exp(-e^{i\varphi})\, d\varphi = 1 + \sum_{\nu=1}^{\infty}\frac{1}{2\pi}\frac{(-1)^\nu}{\nu!}\int_0^{2\pi} e^{\nu i\varphi}\, d\varphi = 1$$

we get for this integral the value

$$\exp\left\{\frac{it\log^2 m}{2\log^{\frac{3}{2}} n} - \frac{t^2\log^3 m}{6\log^3 n}\right\} + O\left\{\frac{1}{\sqrt{\log n}}\right\} + O\left(\frac{1}{m}\right). \tag{10. 1}$$

This and (9. 6) give for $m \leqq n$

$$e_m = \exp\left\{\frac{it\log^2 m}{2\log^{\frac{3}{2}} n} - \frac{t^2\log^3 m}{6\log^3 n}\right\} + O\left(\frac{1}{m}\right) + O\left(\frac{\log\log n}{\sqrt{\log n}}\right). \tag{10. 2}$$

What we actually need is the case $m = n$

$$e_n = \exp\left\{\frac{it\sqrt{\log n}}{2} - \frac{t^2}{6}\right\} + O\left(\frac{\log\log n}{\sqrt{\log n}}\right) \tag{10. 3}$$

and for $m \leqq n$ the inequality

$$|e_m| < c. \tag{10. 4}$$

11. The next (short) *eigtht step* is the determination of $\varphi_n(t)$ based on the representation (8. 4) and on the inequalities (8. 6), (10. 3) and (10. 4). (8. 4) gives

$$\varphi_n(t) = \left\{e_n + \sum_{m=1}^{n-1} e_{n-m}a_m^{(3)} + a_n^{(3)}\right\}\exp\left\{\frac{-it\sqrt{\log n}}{2}\right\}$$

i.e. for $-t_0 \leqq t \leqq t_0$

$$\left|\varphi_n(t) - e^{-\frac{1}{6}t^2}\right| < \frac{c\log\log n}{\sqrt{\log n}}. \tag{10. 5}$$

The *ninth step* is the application of a classical theorem of the calculus of probability. This gives for the distribution-function $F_n(x)$ in (4. 1) the relation

$$\lim_{n\to\infty} F_n(x) = \sqrt{\frac{3}{2\pi}} \int_{-\infty}^{x} e^{-\frac{3}{2}\lambda^2}\, d\lambda \overset{\text{def}}{=} F(x). \tag{10. 6}$$

The last *tenth step* will be the application of our theorem exposed in (2. 3). We have to compare the distribution $K(n, x)$ in our theorem with $F_n(x)$ and $F(x)$ in (10. 6), respectively. (2. 3) gives at once for all n's and real x's

$$K(n, x) \geqq n!\, F_n(x)$$

i.e. for $n\to\infty$

$$\lim_{n\to\infty} \frac{K(n, x)}{n!} \geqq F(x). \tag{10. 7}$$

On the other hand, fixing x and an arbitrarily small $\varepsilon > 0$ we have for $n > n_0(\varepsilon, x)$ the inequality

$$\exp\left\{\frac{1}{2}\log^2 n + x\log^{\frac{3}{2}} n + 3\log n\,(\log\log n)^4\right\} < \exp\left\{\frac{1}{2}\log^2 n + (x+\varepsilon)\log^{\frac{3}{2}} n\right\}$$

i.e. using again (2. 3)

$$n!\,(1 - F_n(x+\varepsilon)) < \varepsilon\, n! + (n! - K(n, x)).$$

Hence for $n\to\infty$

$$\lim_{n\to\infty} \frac{K(n, x)}{n!} \leqq \varepsilon + F(x+\varepsilon). \tag{10. 8}$$

Since ε was arbitrarily small positive, the theorem is proved.

(Received 8 July 1966)