ESSENTIAL HAUSDORFF CORES OF SEQUENCES*

By

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1. Introduction. In this paper we study several concepts concerning families of Toeplitz transformations, with special reference to the Hausdorff methods.

In Sections 2 to 7 we refine Knopp's concept of the core of a sequence of complex numbers, describe the method of essential cores (which extends the idea of Cesàro, Hölder, and Euler methods of infinite order), and compare the method of essential Hausdorff cores with Agnew's collective Hausdorff method.

Section 8 is devoted to the idea of families of matrices that are strong in the sense that the corresponding method of essential cores is stronger than the collective method. In Section 9 we study the norms of bounded convergence fields, and in Section 10 we identify the bounded sequences that are invariant (modulo the addition of a nullsequence) under all regular Hausdorff transformations.

2. The disk-shaped plane. The concept of the core (*Kern*) of a sequence $x=\{x_n\}$ of complex numbers is due to Knopp [15, pp. 113-114], whose definition is equivalent to the following: the core of $\{x_n\}$ is the intersection of all closed convex sets in the extended plane that contain all except finitely many of the points x_n.

To refine the concept, we compactify the plane E^2 by adjoining to it not a single point at infinity, but a circle Γ of points (∞, θ) $(0\leqslant\theta<2\pi)$. In order to topologize our extended plane, we map E^2 onto the tangent hemisphere

$$x^2+y^2+(z-1)^2=1 \qquad (0\leqslant z<1)$$

by the central projection P_1 with centre $(0, 0, 1)$, and we extend the mapping to $E^2 \cup \Gamma$ by adjoining the definition

$$P_1(\infty, \theta)=(\cos\theta, \sin\theta, 1).$$

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We then map the closure of the hemisphere onto the closure of the unit disk by the vertical projection $P_2(x, y, z) = (x, y)$, and declare the topology of the *disk-shaped plane* $E^2 \cup \Gamma$ to be that which the ordinary topology on the closure of D induces under the inverse mapping $P_1^{-1}P_2^{-1}$. For a description of the manner in which P_2P_1 carries each family of parallel lines in $E^2 \cup \Gamma$ onto a family of semi-ellipses with a common diameter, we refer the reader to Gans [9].

By a *half-plane* in $E^2 \cup \Gamma$ we mean one of the two open sets into which the closure of a straight line separates $E^2 \cup \Gamma$; a *closed half-plane* is the closure of a half-plane. It is natural to define convexity in terms of the idea of betweenness. We shall say that a point r in $E^2 \cup \Gamma$ lies *between* two points p and q provided it lies in every closed half-plane containing both p and q. (If in our definition we had used *open* instead of *closed* half-planes, then every point would lie between the points $(\infty, 0)$ and (∞, π), since no open half-plane contains both of these points). We shall further say that a point set E is *convex* provided, whenever (p, q) is a pair of its points, E contains all points lying between p and q.

The set consisting of the two points $(\infty, 0)$ and (∞, π) is convex, but not connected. Clearly, it becomes convex and connected if we adjoin to it a horizontal straight line or the upper or lower semicircle at infinity. Therefore, if the core of a sequence $\{x_n\}$ of finite complex numbers is to be connected in every case, we cannot define it as the intersection of all *connected convex* sets containing all except finitely many of the points x_n, let alone as the intersection of all *convex* sets with this property. This difficulty motivates the following approach.

DEFINITION. If $x = \{x_n\}$ is a sequence of points in the disk-shaped plane, a point p lies in the *core* $K(x)$ provided each half-plane containing p contains x_n for infinitely many n.

If the sequence x is bounded, then $K(x)$ coincides with Knopp's core. To see that the situation is not the same in the general case, let

$$x_n = (-1)^n n,$$
$$y_n = (-1)^n n^2 + in,$$
$$z_n = \begin{cases} n^2 + in & (n \text{ even}), \\ -n^2 & (n \text{ odd}), \end{cases}$$
$$w_n = (-1)^n (n^2 + in).$$

We note that for each of the four sequences, the set of limit points consists of the two points $(\infty, 0)$ and (∞, π). The cores are as follows

$K(x)$ is the closure of the real axis,

$K(y)$ is the closure of the upper half of Γ,

$K(z)$ is the closure of the upper half-plane,

$K(w)$ is $E^2 \cup \Gamma$.

The four cores in the sense of Knopp are

(*i*) the closure of the real axis,

(*ii*) the point at infinity,

(*iii*) the closure of the upper half-plane,

(*iv*) the extended plane.

In (*i*) and (*iii*), the operation of forming the closure is of course carried out under the topology of the extended plane.

An extension of E^2 with a topology finer than that of the disk-shaped plane was described by Rogers [23, Part I, Section 3].

3. **Core-shrinking transformations.** We shall use the symbols $A, B, \ldots$ to denote either Toeplitz matrics $(a_{nk}), (b_{nk}), \ldots$ or the corresponding sequence-to-sequence transformations

$$t = As \quad (t_n = \sum_{k=0}^{\infty} a_{nk}\, s_k).$$

A matrix is *row-finite* if each of its rows contains only finitely many nonzero elements ; otherwise it is *row-infinite*. Also, a matrix (or transformation) A is *regular* (in some terminologies : *permanent*) if convergence of a sequence s to a finite value σ implies the existence of the transform As together with its convergence to σ. For convenience, we state the classical necessary and sufficient conditions for regularity that were established (with varying degrees of generality) by Toeplitz [27, pp. 114-117], Silverman [26, pp. 48-50], Schur [25, pp. 82-88], and others : a matrix A is regular if and only if

$$\text{(3.1)} \qquad \sup_n \sum_{k=0}^{\infty} |\, a_{nk} | < \infty,$$

$$\text{(3.2)} \qquad \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1,$$

$$\text{(3.3)} \qquad \lim_{n\to\infty} a_{nk} = 0 \qquad (k = 0, 1, \ldots).$$

We do not list the many combinations of somewhat weaker or stronger forms of conditions (3.1) to (3.3) that during the last half-century have served as subjects of research papers. Suffice it so say, here, that a row-finite matrix A represents a *core-shrinking* transformation in the space of all complex sequences, in other words, that $K(As) \subset K(s)$ (inclusion in the wide sense) for each sequence s, if and only if A is regular and $a_{nk} \geqslant 0$ for $k > k_o$ (k_o independent of n). For future reference, we state this in a slightly different form :

THEOREM 1. *A row-finite matrix A is core-shrinking if and only if it is regular and can be written as a matrix sum $A = B + C$, where all elements of B are real and nonnegative, and where only finitely many columns of C contain nonzero elements.*

For the case where the core is understood to be Knopp's *Kern*, the proof of the sufficiency of the condition was given essentially by Knopp [15, pp. 115-117], the proof of the necessity by Hurwitz [14] and Agnew [1, p. 185]. We omit the proof of our extension.

If we drop the restriction that A be row-finite, the situation remains simple provided we limit the discussion to bounded sequences. *A matrix A satisfies the condition $K(As) \subset K(s)$ for each bounded sequence s if and only if it is regular and*

$$\lim_{n\to\infty} \sum_k | a_{nk} | = 1.$$

4. Hypercores. If the matrix A is not row-finite, there exist sequences s for which one or more of the series $\Sigma_k a_{nk} s_k$ fail to converge, so that the transform As is not defined. This difficulty can be overcome by defining the point set $K(As)$ directly, that is, without reference to the transformed sequence. For each pair of nonnegative integers n and k, let

$$t_{nk} = \sum_{j=0}^{k} a_{nj} s_j ;$$

for each n, construct the core K_n of the sequence $\left\{ t_{nk} \right\}_{k=0}^{\infty}$; and define the set $\widetilde{K}(A, s)$ as follows : a point in the disk-shaped plane belongs to $\widetilde{K}(A, s)$ provided each half-plane containing the point meets infinitely many of the cores K_n.

For the Euclidean plane, Rogers [24, p. 332] defined the set

$\tilde{K}(A, s)$ and called it the core of the A-transform of s. Since $\tilde{K}(A, s)$ is defined even when As is not defined as a sequence of points, we prefer to call it a *hypercore*.

We shall say that a Toeplitz transformation is *completely regular* in a space S of sequences provided the inclusion relation $\tilde{K}(A, s) \subset K(s)$ holds for each sequence s belonging to S. Clearly, a row-finite matrix is completely regular if and only if it is core-shrinking.

The theorem of Agnew, Hurwitz, and Knopp that we quoted in Section 3 cannot be transferred to hypercores in the obvious manner (for a simple counterexample, see Rogers [24, p. 333]). The following theorem simplifies the problem, but does not solve it. (In form, our theorem resembles Theorem 1 of Kuttner [16]; but since Kuttner's cores are sets in the finite plane, his theorems do not apply here.)

THEOREM 2. *A Toeplitz matrix A is completely regular in the space of complex sequences if and only if*

(*i*) *it has no sequence of nonreal elements $a(n_p, k_p)$ such that both $n_p \to \infty$ and $k_p \to \infty$ as $p \to \infty$;*

(*ii*) *it is completely regular in the space of positive sequences.*

We omit the proof of the sufficiency because it is fairly obvious, the proof of the necessity because it is tedious. However, we call attention to a special class of matrices : suppose A satisfies condition (*i*) in Theorem 2 but has infinitely many rows each of which contains infinitely many negative elements; then A is not completely regular ; for if $s_n \uparrow \infty$ rapidly enough, then the hypercore $\tilde{K}(A, s)$ contains the point (∞, π).

5. Essential cores. Let $\mathfrak{T}$ be a family of completely regular, row-finite transformations that commute with each other under matrix multiplication. Then, for each pair of matrices A and B in $\mathfrak{T}$ and each sequence s of complex numbers, it follows from the two relations

$$K(As) \supset K(BAs) \quad \text{and} \quad K(Bs) \supset K(ABs)$$

that the intersection $K(As) \cap K(Bs)$ is not empty. Since the cores are closed sets and the disk-shaped plane is compact, it follows further that for each sequence s the set

$$K(\mathfrak{T}, s) = \bigcap_{A \in \mathfrak{T}} K(As) \tag{5.1}$$

contains at least one point.

The requirement that the family $\mathfrak{T}$ be a commutative semigroup under multiplication can be replaced with the weaker condition that to each pair A and B of transformations in $\mathfrak{T}$ and each sequence t there correspond a tranformation $C = C(A, B, t)$ in $\mathfrak{T}$ such that

$$K(At) \supset K(Ct) \quad \text{and} \quad K(Bt) \supset K(Ct).$$

If $\mathfrak{T}$ is a family of completely regular, row-finite matrices satisfying this condition, we call the set (5.1) the *essential $\mathfrak{T}$-core of s*. By the method of Knopp [15, pp. 115-117], it can be proved that the essential $\mathfrak{T}$-core of a sequence is connected and convex in the disk-shaped plane.

THEOREM 3. *Let A and B be two row-finite, completely regular transformations such that AB = BA, and let*

$$\mathfrak{S} = \{I, A, A^2, \ldots\},\ \mathfrak{T} = \{I, B, B^2, \ldots\},\ \mathfrak{U} = \{I, AB, (AB)^2 \ldots\}.$$

Then, for each sequence s,

$$K(\mathfrak{S}, s) \cap K(\mathfrak{T}, s) \supset K(\mathfrak{U}, s).$$

The inclusion symbol cannot in general be reversed.

PROOF. If p is a point in $K(\mathfrak{U}, s)$, then p lies in $K\{(AB)^n s\}$, for each n. Since A and B are completely regular and $(AB)^n = A^n B^n = B^n A^n$, it follows that p lies in $K(A^n s)$ and in $K(B^n s)$, for each n. This proves the inclusion relation.

To complete the proof, let

$$A = \begin{pmatrix} \alpha & & & & & \\ & \beta & & & & \\ & & \alpha & & & \\ & & & \beta & & \\ & & & & \cdot & \\ & & & & & \cdot \end{pmatrix}, \quad B = \begin{pmatrix} \beta & & & & & \\ & \alpha & & & & \\ & & \beta & & & \\ & & & \alpha & & \\ & & & & \cdot & \\ & & & & & \cdot \end{pmatrix},$$

where $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Then

$$AB = BA = \begin{pmatrix} \beta & & & & \\ & \beta & & & \\ & & \beta & & \\ & & & \cdot & \\ & & & & \cdot \end{pmatrix}.$$

Now let $s_k = (-1)^k \ (k = 0, 1, \ldots)$. Then for $n = 1, 2, \ldots$,

$$A^n s = \{1, -1, 0, 0, 1, -1, 0, 0, \ldots\},$$
$$B^n s = \{0, 0, 1, -1, 0, 0, 1, -1, \ldots\},$$
$$(AB)^n s = \{0, 0, 0, \ldots\}.$$

Therefore the inclusion relation cannot be reversed, and the theorem is proved.

6. Completely regular Hausdorff transformations. A Toeplitz matrix A is a *Hausdorff matrix* provided it has the form

$$A = \delta \, \mu \, \delta, \tag{6.1}$$

where μ is a diagonal matrix and δ denotes the self-reciprocal triangular matrix

$$\begin{pmatrix} 1 & & & & \\ 1 & -1 & & & \\ 1 & -2 & 1 & & \\ 1 & -3 & 3 & -1 & \\ 1 & -4 & 6 & -4 & 1 \\ & \cdots & \cdots & \cdots & \end{pmatrix}$$

(see Hausdorff [13] or Hardy [11, Chapter 11]). From the definition it follows immediately that the Hausdorff matrices form an abelian semigroup and that the Hausdorff matrices without 0's on the diagonal form a group.

If $A = (a_{nk})$ is a Hausdorff matrix, then (3.1) holds if and only if there exists a function $\alpha(u)$, of bounded variation on $[0, 1]$, such that

$$a_{nn} = \int_0^1 u^n \, d\alpha(u) \qquad (n = 0, 1 \ldots). \tag{6.2}$$

In cases where the function α exists, we shall always assume that $\alpha(0) = 0$ and that

$$2\,\alpha(u) = \alpha(u + 0) + \alpha(u - 0) \qquad (0 < u < 1).$$

With this restriction, α becomes unique, and we shall call it the *generating function* of A (notation : $A = H\{\alpha\}$).

It follows from (6.1) and (6.2) that if A is a Hausdorff matrix with a generating function α, then

$$a_{nk} = \binom{n}{k} \int_0^1 u^k (1 - u)^{n-k} \, d\alpha(u) \tag{6.3}$$

for $n = 0, 1, \ldots$ and $k = 0, 1, \ldots, n$. Formula (6.3) in turn implies that if $0 \leqslant u < v \leqslant 1$, then

$$\lim_{n \to \infty} \sum_{un \leqslant k \leqslant vn} a_{nk} = \alpha(v) - \alpha(u). \tag{6.4}$$

Moreover, if M is a closed set on $[0, 1]$ on which α is continuous, then (6.4) holds uniformly as long as u and v are restricted to M.

If α and β generate two Hausdorff transformations A and B, then the matrix product AB is generated by the function

$$(6.5) \qquad \gamma(u) = \beta(1)\,\alpha(u) + \int_u^1 \beta(u/v)\, d\alpha(v)$$

(certain conventions concerning the definition of the Stieltjes integral must be invoked, in case $\beta(u/v)$ and $\alpha(v)$ have common points of discontinuity).

The right member of (6.5) is not generally suitable for computations; therefore early work on Hausdorff transformations, which was concerned largely with the comparison of individual transformations, depended mainly on the analysis of the sequence $\{a_{nn}\}$, that is, of the diagonal matrix μ in (6.1). For our purposes, careful computations will not be important; on the other hand, we shall require intuitive insight into the relation between the generating function α and the matrix $H\{\alpha\}$, and therefore we shall rely mainly on (6.4).

From (6.3) we see immediately that

$$\lim_{n\to\infty} a_{n0} = \lim_{n\to\infty} \int_0^1 (1-u)^n\, d\alpha(u) = \lim_{u\to 0} \alpha(u).$$

Together with (6.4), this implies that $H\{\alpha\}$ is regular if and only if $\alpha(1) = 1$ and α is continuous at $u = 0$. Moreover : *a regular Hausdorff matrix is completely regular if and only if its generating function is real and monotonic.*

7. Collective methods and essential Hausdorff cores. We write $\mathfrak{H}$ for the family of regular Hausdorff transformations, and $\mathfrak{H}^+$ for the family of completely regular Hausdorff transformations. A sequence s of complex numbers is said to be summable to the (finite) complex value σ by the *collective Hausdorff method* $\{\mathcal{H}\}$ provided there exists a matrix H in $\mathfrak{H}$ such that $Hs \to \sigma$; if τ is a point of the disk-shaped plane, we say that s is summable to τ by the *method* $K\mathfrak{H}^+$ *of essential Hausdorff cores* provided the essential core $K(\mathfrak{H}^+, s)$ consists of the point τ. The distinction : for each sequence s the collective method

$\{\mathcal{H}\}$ polls all regular Hausdorff transformations, but in each case records the reply only if the transform of s converges to a finite point; the essential method $K\mathfrak{H}^+$ consults only the *completely* regular Hausdorff transformations, but ignores none of the opinions that it obtains. Theorems 4 to 6 deal with the question of consistency and relative strength of the two methods.

Zeller [29, p. 11] calls collective methods and essential-core methods *Vereinigungsverfahren* and *Einschachtelungsverfahren*, respectively. The collective Cesàro method was first discussed by Hardy and Littlewood [12, p. 67]. Agnew called attention to the collective Euler method [3, p. 321]; earlier [2, Section 6], he had already investigated the collective Hausdorff method, the first collective method not obviously equivalent to some collective method based on a sequence of progressively stronger matrix methods. Fuchs showed [8] that no matrix method contains Agnew's collective Hausdorff method, and raised the question whether some matrix method contains the collective method of "reasonable" Hausdorff matrices.

Meyer-König and Zeller [19] proved that no countable collective method is equivalent to a matrix method A unless one of its contributing methods is equivalent to A. This result can not in general be extended to an uncountable collection. For example, let A denote the method that transforms $\{s_n\}$ into $\{(s_{n-1}+s_n)/2\}$. Corresponding to each sequence s summed to 0 by A, we construct a matrix A_s such that A_s sums s and is weaker than A but consistent with A. This can be done (in the spirit of the proof of Theorem 2 of [7]) by adjoining to A infinitely many rows that contain exactly two nonzero elements lying in widely separated columns. The collective method determined by the family $\{A_s\}$ is clearly equivalent to the method A.

Eberlein [6] extended the concept of the collective Hausdorff method to the concept of the *Banach-Hausdorff limit*. References to other early papers on essential-core methods will be made in Section 8.

THEOREM 4. *There exists a sequence s that is summed to* 0 *by the collective Hausdorff method* $\{\mathcal{H}\}$, *and to the point* $(\infty, 0)$ *of the disk-shaped plane by the method* $K\mathfrak{H}^+$ *of essential Hausdorff cores.*

Let $s_n = n$ $(n = 0, 1 \ldots)$, and let $A=2C-I$, where C is the method of arithmetic means (that is, the Hausdorff matrix generated by the function $\alpha(u) = u$), and where I is the identity matrix. If $As = t$, then

$$t_n = 2\,(1 + 2 + \ldots + n)/(n + 1) - n = 0$$

for all n, and therefore, the method $\{\mathcal{H}\}$ sums s to 0. Since the core $K(s)$ consists of the point $(\infty, 0)$, the method $K\mathfrak{H}^+$ sums s to $(\infty, 0)$.

PROBLEM 1. Does there exist a sequence s that is summable by Agnew's collective Hausdorff method $\{\mathcal{H}\}$ and whose essential Hausdorff core $K(\mathfrak{H}^+, s)$ contains more than one point?

DEFINITION. Let $\alpha(u)$ be a real-valued function of bounded variation on $[0, 1]$, with $\alpha(0) = 0 < \alpha(1)$, and let A denote the Hausdorff matrix $H\{\alpha\}$. Let α_+ and α_- denote the least nondecreasing functions, with $\alpha_+(0) = \alpha_-(0) = 0$, for which $\alpha(u) = \alpha_+(u) - \alpha_-(u)$, and let

$$\tilde{\alpha}(u) = \alpha_+(u)/\alpha_+(1).$$

We shall call the Hausdorff matrix $A_+ = H\{\tilde{\alpha}\}$ the *normalized positive part of A*, and we shall use the symbol $\mathfrak{A}_+$ for the family $\{I, A_+, A_+^2, \ldots\}$.

THEOREM 5. *If A is a real, regular Hausdorff matrix and s is a bounded sequence such that* $As \to 0$, *then* $K(\mathfrak{A}_+, s) = \{0\}$.

Our proof was suggested by Eberlein in a private communication. Suppose that the essential core $K(\mathfrak{A}_+, s)$ contains a point $re^{i\theta}$ $(r > 0)$. It is to be shown that As does not tend to 0.

Let $\alpha = \alpha_+ - \alpha_-$ be the decomposition mentioned above, and let $B = H\{\alpha_+\}$, $C = H\{\alpha_-\}$. For convenience, we write $\limsup |s_n| = \| s \|$ ($\| \|$ is then not actually a norm on the space of bounded sequences; but if we define two sequences to be equivalent if their difference is a null-sequence, then $\| \|$ is a norm on the space of equivalence classes of bounded sequences).

The hypothesis on $K(\mathfrak{A}_+, s)$ implies that $\| A_+{}^p s \| \geqslant r$ for $p = 1, 2, \ldots$, in other words, that

$$\| B^p s \| = [\alpha_+(1)]^p \, \| A_+^p s \| \geqslant r \, [\alpha_+(1)]^p.$$

Together with the relation (6.4), the fact that C is real and non-negative implies that

$$\| C^p s \| \leqslant \| s \| \, [\alpha_-(1)]^p.$$

Since $\alpha_+(1) = \alpha - (1) + 1 > \alpha_-(1)$, since $\| (B^p - C^p) s \| > 0$ for all p for which

$$r \, [\alpha_+(1)]^p > \| s \| \, [\alpha_-(1)]^p,$$

and since the left factor in the right-hand member of the matrix identity

$$B^p - C^p = (B^{p-1} + B^{p-2} C + \ldots + C^{p-1})(B - C)$$

transforms every nullsequence into a nullsequence, it follows that $\|(B - C) s\| > 0$, as was to be proved.

THEOREM 6. *In the space of sequences whose essential Hausdorff core is bounded, the method of essential Hausdroff cores is stronger than the collective Hausdorff method* $\{\mathcal{H}\}$, *and it is consistent with it.*

PROOF. Let s be a sequence for which the set $K(\mathfrak{H}^+, s)$ is bounded, and let U denote a bounded open set that contains $K(\mathfrak{H}^+, s)$. To each point q in the complement of U there corresponds a matrix T in $\mathfrak{H}^+$ such that q lies in the complement of the core $K(Ts)$. By the theorem of Borel and Lebesgue, there exists a finite set of matrices $B_1, B_2, \ldots, B_j$ in $\mathfrak{H}^+$ such that the union of the complement of the cores $K(B_i s)$ $(i = 1, 2, \ldots, j)$ contains the complement of U. Define B to be the matrix product $B_1 B_2 \ldots B_j$; then $K(Bs)$ is contained in U and is therefore bounded.

Now suppose that C is a matrix in $\mathfrak{H}$ such that Cs converges to a finite value p ; let $\overline{C}$ denote the complex conjugate of C ; let A_+ denote the normalized positive part of the real matrix $A = C\overline{C}$; and let $\mathfrak{A}_+$ be the family of powers of A_+. Since ABs converges to p, it follows from Theorem 5 that $K(\mathfrak{A}_+, Bs) = \{p\}$, and since B is completely regular, we conclude that the method $K\mathfrak{H}^+$ is at least as strong as $\{\mathcal{H}\}$ and is consistent with $\{\mathcal{H}\}$, in the space under consideration.

It remains to exhibit a sequence s that is not summable by any regular Hausdorff method and whose essential Hausdorff core consists of precisely one finite point. For this purpose we choose any auxiliary sequence $\{x_m\}$ $(m = 1, 2, \ldots)$ that is dense on the interval $(0, 1)$, and any sequence of positive integers p_m with the property

$$(7.1) \qquad 1 < x_m\, p_m / p_{m-1} \to \infty.$$

We define our sequence s by the rule that $s_n = x_m$ if $x_m\, p_m \leqslant n \leqslant p_m$ and that $s_n = 0$ whenever n lies in none of the intervals $[x_m\, p_m, p_m]$.

Corresponding to each number t $(0 < t < 1)$ we define the function

$$\alpha_t(u) = \begin{cases} 0 & (u = 0), \\ \max(0, 1 + t \log u) & (0 < u \leqslant 1), \end{cases}$$

and we write $A_t = H\{\alpha_t\}$. We observe that $\alpha_t(0) = 0$ in the interval $0 \leqslant u \leqslant e^{-1/t}$, that $\alpha_t(u)$ increases in $e^{-1/t} \leqslant u \leqslant 1$, and that

$$(7.2) \qquad \alpha_t(u) - \alpha_t(xu) \leqslant t \log 1/x \qquad (0 < x \leqslant 1),$$

the equality sign being in force whenever $xu \geqslant e^{-1/t}$.

Consider now the transform $A_t s$, for a fixed value of the parameter t. By (7.1), there exist at most finitely many indices n such that more than one of the blocks of consecutive positive elements s_k have indices falling into the range $ne^{-1/t} < k \leqslant n$. Therefore (6.4) and (7.2) imply that

$$\| A_t s \| = \limsup_{m\to\infty} \{ \sup_{0 < u \leqslant 1} x_m [\alpha_t(u) - \alpha_t(x_m u)] \}$$

$$\leqslant \limsup_{m\to\infty} t x_m \log 1/x_m \leqslant t/e.$$

It follows that $\lim_{t\to 0} \| A_t s \| = 0$, and the method $K\mathfrak{H}^+$ sums s to 0.

Suppose, on the other hand, that $H\{\beta\}$ is a Hausdorff matrix that satisfies (3.1) and sums the sequence s. Because $s_k = 0$ for $p_{m-1} < k < x_m p_m$, it follows from (7.1) that the transform of s can converge only to 0. Because by (6.4) and (7.1) the p_mth element of the transform is

$$x_m [\beta(1) - \beta(x_m)] + o(1),$$

it follows that $\beta(x_m) = \beta(1)$. Since the sequence $\{x_m\}$ is dense in $(0,1)$, it follows further that $\beta(u) = \beta(1)$ for $0 < u \leqslant 1$. Consequently, if β is continuous at 0, then $\beta(u) = 0$, and therefore $H\{\beta\}$ is not regular. Hence s is not summable by the method $\{\mathcal{H}\}$, and the proof is complete.

We return briefly to Theorem 5, which can be paraphrased as follows: If A is a real, regular Hausdorff matrix, then, *in terms of the method of essential cores*, the sequence of powers of the normalized positive part of A is at least as effective as A itself, in the space of bounded sequences. The following theorem shows that the statement becomes false when it is interpreted *in terms of the collective method.*

THEOREM 7. *There exist a real regular Hausdorff matrix A and a bounded sequence s such that $As \to 0$ but $\overset{p}{A_+} s$* does not tend to 0 $(p = 1, 2, \ldots)$.

PROOF. Let α be constant except for salti 1, 1/2, 1/2, and -1 at $u = 1/4, 1/3, 2/3$, and 1, respectively; let $s_o = 0$ and

$$s_n = \sin(\pi \log_2 n) \quad (n = 1, 2, \ldots),$$

and write $As = t$. Let ε be any positive number less than 1/48, and for each n greater than 48, let $k_1 = k_1(n), \ldots, k_6 = k_6(n)$ denote the integral parts of the numbers

$$(1/4 - \varepsilon)n,\ (1/4 + \varepsilon)n,\ (1/3 - \varepsilon)n, \ldots,\ (2/3 + \varepsilon)n,$$

Then, by (6.4),

$$t_n = \Big(\sum_{k_1}^{k_2} + \sum_{k_3}^{k_4} + \sum^{k_6}\Big) a_{nk}\, s_k - s_n + o\,(1)$$

$$= \sin\Big(\pi \log_2 \frac{n}{4}\Big) + \frac{1}{2}\sin\Big(\pi \log_2 \frac{n}{3}\Big) + \frac{1}{2}\sin\Big(\pi \log_2 \frac{2n}{3}\Big)$$

$$- \sin(\pi \log_2 n) + O(\varepsilon) + o(1)$$

$$= O(\varepsilon).$$

Since ε is arbitrary, $As \to 0$.

Now write $A_+ s = \tau$. Then

$$\tau_n = \frac{1}{2}\Big(\sum_{k_1}^{k_2} + \sum_{k_3}^{k_4} + \sum_{k_5}^{k_6}\Big) a_{nk}\, s_k + o\,(1)$$

$$= \frac{1}{2}\sin\Big(\pi \log_2 \frac{n}{4}\Big) + o\,(\varepsilon),$$

and therefore $\tau_n = s_n/2 + o\,(1)$. It follows that $A_+^p s = 2^{-p}\, s + c^{(p)}$, where $c^{(p)}$ is a nullsequence. This concludes the proof.

PROBLEM 2. Does there exist a countable set $\mathfrak{A} = \{A_k\}$ of completely regular Hausdorff transformations such that every sequence summable by some regular Hausdorff method is summable by the method of essential $\mathfrak{A}$-cores ?

8. **Strong and weak families**. Again, let $\mathfrak{T}$ be a family of completely regular, row-finite matrices satisfying the conditions in the second paragraph of Section 5, and let S be a space of sequences. We say that $\mathfrak{T}$ is a *strong family* (in S) if S contains a sequence s with the property that the essential $\mathfrak{T}$-core $K(\mathfrak{T}, s)$ consists of a single point in the disk-shaped plane while for each T in $\mathfrak{T}$ the core $K(Ts)$ contains more than one point; if $\mathfrak{T}$ is not strong, it is *weak*. In other words, the family $\mathfrak{T}$ is strong provided the method of essential $\mathfrak{T}$-cores is stronger than the collective $\mathfrak{T}$-method. The families of transformations A_t and A^p used in the proofs of Theorems 6 and 7 are both strong, even in the space of bounded sequences. We shall now consider certain other families.

Among the best-known Hausdorff matrices are the Hölder and the Cesàro matrices. The Hölder matrix H_k of order k is the Hausdorff matrix with the diagonal elements

$$a_{nn} = (n+1)^{-k} \qquad (n = 0, 1, \ldots)$$

(if k is complex, the determination is made unambiguous by the convention that $1^k = 1$ and by the requirement of continuity in the right half of the n-plane). The matrix H_k is regular if and only if $k = 0$ or $\Re k > 0$. In the latter case, H_k is generated (see [13, p. 88]) by the function

$$a_k(u) = \frac{1}{\Gamma(k)} \int_0^u |\log v|^{k-1} \, dv.$$

The Cesàro matrix C_k of order k is the Hausdorff matrix with the diagonal elements

$$b_{nn} = 1 \Big/ \binom{n+k}{k}.$$

It is regular provided $k = 0$ or $\Re k > 0$, and in the latter case, it has the generating function

$$\beta_k(u) = 1 - (1-u)^k.$$

We shall restrict the discussion to the cases where H_k and C_k are completely regular, that is, to the cases where $k \geqslant 0$.

The methods H_k and C_k are equivalent with respect to summability to finite values: for each sequence s and each index k, either both or neither of the two transforms $H_k s$ and $C_k s$ converges to a finite value [13, p. 88]. However, Garten and Knopp have shown [10, Section 5] that in the space of real sequences the method of essential Hölder cores is more effective than the method of essential Cesàro cores. The basic difference between the two methods becomes apparent if we examine the corresponding families of generating functions and consider equation (6.4): For each fixed number r $(0 < r < 1)$ and for $k > 1$, it is obvious that

$$\alpha_k(u) - \alpha_k(ru) = \frac{1}{\Gamma(k)} \int_{ru}^{u} |\log v|^{k-1} \, dv < \frac{(1-r)u}{\Gamma(k)} |\log ru|^{k-1}.$$

For each $k > 1$, the maximum value of the last member is

$$\frac{1 - r}{r\Gamma(k)} [(k - 1)/e]^{k-1},$$

and by Stirling's formula, this tends to 0 as $k \to \infty$.

On the other hand,

$$\beta_k(u) - \beta_k(ru) = (1 - ru)^k - (1 - u)^k,$$

and for the special choice $u = 1/k$ this tends to $e^{-r} - e^{-1}$, as $k \to \infty$. Clearly, the sequence s used in the proof of Theorem 6 is summable by the method of essential Hölder cores, but not by the method of essential Cesàro cores, and therefore not by any individual Hölder transformation. This proves the following proposition.

THEOREM 8. *The family of Hölder transformations is strong in the space of bounded sequences.*

If we examine again the generating functions of the matrices A_t used in the proof of Theorem 6, we note that the family of Hölder matrices has a high digestive capacity in the space of bounded sequences not because the graphs of its generating functions have a vertical half-tangent at $u = 0$, for $k > 1$, but because, for each r, $\alpha_k(u) - \alpha_k(ru) \to 0$ uniformly with respect to u, as $k \to \infty$. We observe further that the effectiveness of the method of essential Hölder cores does not depend on the fact that the method of arithmetic means is fairly powerful. Indeed, let λ denote a constant $(0 < \lambda < 1)$, let M be the Hausdorff method generated by the function α with

$$\alpha(u) = \lambda u \qquad (0 \leqslant u < 1), \qquad \alpha(1) = 1,$$

and let $\mathfrak{M}$ be the family of powers of M. The sequence Ms diverges whenever s diverges, by a theorem of Mercer [18, Theorem I], [11, pp. 104, 106-107]. On the other hand, it is easy to verify that if s is a bounded sequence, then the essential $\mathfrak{M}$-core of s coincides with the essential Hölder-core of s. In particular, the method of essential $\mathfrak{M}$-cores sums the sequence used in the proof of Theorem 6, and therefore the family $\mathfrak{M}$ is strong.

It has long been known that the family of completely regular Cesàro transformations is weak in the space of sequences whose essential Cesàro core is bounded; but the proof is deep: it depends on the comparison of the essential Cesàro core with the Abel core. [If $\limsup |s_n|^{1/n} \leqslant 1$, the Abel transform of s is the function

$$f(x) = (1 - x) \sum s_n x^n.$$

The *Abel core* of s is the core of $f(x)$, defined (with respect to the behaviour of $f(x)$ as $x \to 1$ by an obvious generalisation of the definition of cores.) The following striking proposition was first proved by Ramaswami [22] (Ramaswami dealt only with real sequences ; but the restriction is unimportant, and we omit the proof of the generalization).

Theorem 9 (Ramaswami). *If a sequence s has a bounded Abel core, then its essential Cesàro core consists of*

(*i*) *the Abel core, or*

(*ii*) *an infinite strip that contains the Abel core and is bounded by two parallel lines of support of the Abel core, or*

(*iii*) *the disk-shaped plane.*

A sequence s cannot have a bounded essential Cesàro core unless one of its Cesàro transforms is bounded. But as Littlewood pointed out [17], if $K(C_k s)$ is bounded and the Abel core of s consists of one point, then $C_h s$ converges for $h > k$, and therefore the method of essential Cesàro cores is weak in the space of sequences whose essential Cesàro core is bounded.

That the family of Cesàro transformations is strong in the space of all sequences can be seen from the following example : Let $n_1 \geqslant 2$, $n_{r+1} \geqslant n_r^{r+1}$, and choose

$$s_n = \begin{cases} n_{r+1} & \text{if } n = n_r \quad (r = 1, 2, \ldots), \\ 0 & \text{otherwise.} \end{cases}$$

Here the core of $C_k s$ consists of the segment $[\, k, \infty \,]$ on the real axis ; we omit the computations.

The regular Euler matrices are the Hausdorff matrices E_x generated by the functions

$$\alpha_x(u) = \begin{cases} 0 & (0 \leqslant u < x), \\ 1 & (x < u \leqslant 1), \end{cases}$$

with $0 < x \leqslant 1$. They satisfy the relation $E_x E_y = E_{xy}$. Meyer-König and Zeller [19] have shown that the family of Euler methods is weak in the space of sequences whose essential Euler core is bounded.

Problem 3. To find conditions on a completely regular Hausdorff matrix $A\{x\}$ that are necessary and sufficient for the family

$\{A^n\}$ $(n=1, 2,...)$ to be strong (strong in the space of bounded sequences), and to express these conditions as conditions on the generating function α.

A natural analogue to Problem 3 arises in the theory of Nörlund transformations. To each completely regular Nörlund transformation N corresponds a function $f(z) = \Sigma p_n z^n$ $(p_n \geqslant 0)$ holomorphic in some neighbourhood of the origin. In particular, the Cesàro matrices (the only matrices that are both Hausdorff and Nörlund matrices ; see Ullrich [28] or Agnew [4]) correspond to the functions $f(z) = (1-z)^{-k}$. We are indebted to the referee for the following theorem.

THEOREM 10. *Let $\mathcal{F}$ be a family of polynomials f with $f(1) = 1$ and with positive coefficients, and let $\mathcal{F}$ contain polynomials whose largest coefficient is arbitrarily small. Then the family of Nörlund transformations corresponding to $\mathcal{F}$ is strong in the space of bounded sequences.*

PROOF. Let the sequence s consist of blocks of k elements $1/k$ $(k = 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, ...)$ with a gap of j zeros between the jth and $(j+1)$ st blocks :

$$\{s_n\} = \{1, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 1, 0, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0, 0, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0, \ ...\}$$

If N_f denotes the Nörlund method corresponding to $f \in \mathcal{F}$, and if f has degree k, then the sequence $N_f\, s$ contains infinitely many elements $1/(k+1)$ and infinitely many elements 0. Therefore the collective method corresponding to $\mathcal{F}$ does not sum s. However, $\| N_f s \|$ is not greater than the largest coefficient of f, and therefore the method of essential cores corresponding to $\mathcal{F}$ sums s to 0. This proves the theorem.

9. The norms of bounded convergence fields. Corresponding to a regular Toeplitz matrix A we denote by (A) the *bounded convergence field* of A, that is, the set of all bounded sequences s for which As converges. If A and B are regular Toeplitz matrices and $(A) \supset (B)$, then A and B are consistent in the space of bounded sequences. Brudno [5] observed that therefore a sequence in the bounded convergence field (A) of a regular Toeplitz matrix A can be regarded as being summable not only by the matrix, but by the bounded convergence field itself, in other words, that a bounded convergence field is significant not merely as a collection of sequences associated with a matrix, but as a set that associates with each of its elements s a complex number $\sigma(s)$ determined directly by the set.

Brudno defined the *norm* $\| A \|$ *of a Toeplitz matrix* A as the left member in the relation (3.1), and the *norm* $\| (A) \|$ *of a bounded* convergence field as the infimum of the norms of all regular Toeplitz matrices B such that $(B) = (A)$.

DEFINITION. If (A) is a bounded convergence field, we say that (A) has *accessible norm* provided there exists a regular Toeplitz matrix B such that $(B) = (A)$ and $\| B \| = \| A \|$. If (A) does not have accessible norm, it has *inaccessible norm*.

PROBLEM 4. To characterize the bounded convergence fields of accessible norm, without reference to Toeplitz matrices.

THEOREM 11. *There exist bounded convergence fields of inaccessible norm.*

EXAMPLE. Let c denote any constant ($c \neq 0$), and let

$$A = \begin{pmatrix} 1 & & & & & \\ 1 & c & -c & & & \\ & & & 1 & & \\ & & & 1 & c & -c \\ \multicolumn{6}{c}{\dots\dots\dots\dots\dots\dots} \end{pmatrix},$$

where all unmarked elements in the first four rows are understood to be 0's. Clearly, As converges if and only if s differs by a convergent sequence from a sequence of the form

$$\{0, x_1, x_1, 0, x_2, x_2, 0, x_3, x_3, \dots\}. \tag{9.1}$$

Therefore the convergence field of A is independent of c, and since $\| A \| = 1 + 2 | c |$, it follows that $\| (A) \| = 1$.

It remains to prove that if B is regular and $(B) = (A)$, then $\|B\| > 1$. Without loss of generality we may assume that

$$\lim_{n\to\infty} \sum_k | \mathfrak{J} b_{nk} | = 0;$$

for otherwise $\| B \| > 1$ by (3.2). Since only bounded sequences concern us, we can therefore disregard the imaginary part of B and assume that B is real.

If $(B) = (A)$, then

$$\liminf_{k\to\infty} (\max_n | b_{nk} |) > 0; \tag{9.2}$$

indeed, if (9.2) fails, then there exists a sequence of 0's and 1's, with the 1's arbitrarily far apart, such that $Bs \to 0$, and therefore $(B) \neq (A)$.

Suppose then that (9.2) holds, and that $(B) = (A)$. Then $Bt \to 0$, where $t = \{0, 1, 1, 0, 1, 1, \ldots\}$. This implies that

$$\liminf_{k\to\infty} (\min_n b_{nk}) < 0.$$

Together with condition (3.2), this implies that $\| B \| > 1$, and the proof is complete. (A different example was described by Petersen [21, Theorem 3]).

PROBLEM 5. Let B be a bounded convergence field containing a divergent sequence s. Do there exist bounded convergence fields (A) and (C), of accessible norms, such that $s \in (A) \subset (B) \subset (C)$?

We point out that some bounded convergence fields of inaccessible norm are contained in bounded convergence fields of accessible norm. For example, let S consist of the bounded sequences that have the form (9.1) and meet the further restriction that $x_n = 0$ except when n is a square; and let Q be the set of sequences that differ by a convergent sequence from one of the elements of S. Then Q is a bounded convergence field of inaccessible norm, and since Q is contained in the bounded convergence field of the matrix C_1, our assertion is proved.

THEOREM 12. *If A is a regular Hausdorff matrix, then* $\|(A)\| = 1$.

PROOF. Let B and C represent the matrices used in the proof of Theorem 5, and for $p = 1, 2, \ldots$, let

$$D_p = \frac{B^p - C^p}{[\alpha_+(1)]^p - [\alpha_-(1)]^p}$$

Then $\| D_p \| \to 1$ as $p \to \infty$. Clearly, the convergence field of D_p contains that of the matrix $A = D_1$.

Now let $\varepsilon > 0$, and let E_p denote a matrix consisting of all rows of D_p and all rows of $(1 - \varepsilon) D_p + \varepsilon A$. Then $(E_p) = A$. Since $\| E_p \| \to 1 + (\| A \| - 1)\varepsilon$ as $p \to \infty$, the theorem follows.

(The second part of our proof is based on a method of Brudno [5, Theorem 2]; for an English version, see Petersen [20, Theorem 2]).

10. Invariant sequences. If A is a Toeplitz transformation, we shall say that a sequence s is *invariant under* A provided $As - s$ is a nullsequence; we shall say that s is *invariant under a family of Toeplitz transformations* if it is invariant under each transformation of the family.

THEOREM 13. *A bounded sequence s is invariant under C_1 if and only if, for each $\lambda > 0$,* (10.1) $$\lim_{n\to\infty} \max_{0<p<n\lambda} |s_{n+p} - s_n| = 0.$$

PROOF. The sufficiency of the condition is obvious. In proving the necessity, we assume without loss of generality that s is real.

If (10.1) fails for some λ, then there exist a positive number h an sequences $\{n_i\}$ and $\{p_i\}$, with $n_i \to \infty$ and $0 < p_i < n_i\lambda$, such that

$$|s(n_i + p_i) - s(n_i)| > h \qquad (i = 1, 2, \ldots).$$

We may suppose that for each i the integer p_i is the least natural number p such that $|s(n_i + p) - s(n_i)| > h$, and that for $0 < p < p_i$ the sign of $s(n_i + p) - s(n_i)$ is always the same, say positive. Let $\sigma = C_1 s$, and write $\mu_i = p_i/(n_i + p_i + 1)$. Then

$$\sigma(n_i + p_i) = \frac{1}{n_i + p_i + 1}\left\{ (n_i + 1)\,\sigma(n_i) + \sum_{k=n_i+1}^{n_i+p_i} s_k \right\}$$

$$\leqslant (1 - \mu_i)\,\sigma(n_i) + \mu_i\,[s(n_i) + h] + \frac{s(n_i + p_i) - [s(n_i) + h]}{n_i + p_i + 1},$$

and since the numerator in the last term is a bounded function of the index i, it follows that

$$s(n_i + p_i) - \sigma(n_i + p_i) \geqslant s(n_i) + h - (1 - \mu_i)\sigma(n_i) - \mu_i\,[s(n_i)+h]+o)1)$$

$$= (1 - \mu_i)\,[h + s(n_i) - \sigma(n_i)] + o(1).$$

Consequently, $s(n_i + p_i) - \sigma(n_i + p_i)$ and $s(n_i) - \sigma(n_i)$ cannot both be $o(1)$. This implies that s is not invariant under C_1, and the proof is complete.

THEOREM 14. *If a bounded sequence is invariant under the transformation C_1, it is invariant under the family of regular Hausdorff transformations.*

PROOF. Let $A\{\alpha\}$ denote any regular Hausdorff matrix, let s be a sequence invariant under C_1, and write $t = As$. Then

$$t_n - s_n = \left\{ \sum_{k<un} + \sum_{k\geqslant un} \right\} a_{nk}(s_k - s_n).$$

Let $\varepsilon > 0$. By (6.4) and the regularity of A, we can choose u small enough so that the first sum on the right has absolute value less than ε when n is large.

As $n \to \infty$, the second sum tends to 0, by Theorem 13; this concludes the proof.

We point out that invariance under a specific regular Hausdorff transformation stronger than the identity need not imply invariance under all regular Hausdorff transformations. For example, let $s_n = \sin(\pi \log_2 n)$ for $n = 1, 2, \ldots$, and let A be the regular Hausdorff matrix generated by the function α that is constant except for salti of height 1/2, at $u = 1/4$ and at $u = 1$. Then s is invariant under A. Since s violates condition (10.1), it is not invariant under C_1; hence it is not invariant under all regular Hausdorff transformations.

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