ACTA ARITHMETICA
IX (1964)

The star number of coverings of space with convex bodies

by

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In honour of Professor L. J. Mordell

1. When a system of sets covers a space, the star number of the covering is the supremum over the sets of the system of the cardinals of the numbers of sets of the system meeting a set of the system. The standard Lebesgue 'brick-laying' construction provides an example, for each positive integer n, of a lattice covering of R^n by closed rectangular parallelepipeds with star number $2^{n+1}-1$. In view of the results of dimension theory, it is natural to conjecture that any covering of R^n by closed sets of uniformly bounded diameter has star number at least $2^{n+1}-1$; and this has been proved by V. Boltyanskiĭ [1] in the special case $n = 2$.

In this paper we consider only coverings of R^n by translates of a fixed convex body. We first give a simple proof (the idea of which comes from the work of Minkowski and Voronoï) of

THEOREM 1. *The star number of a lattice covering of R^n by translates of a closed symmetrical convex body is at least $2^{n+1}-1$.*

Then we consider the problem of constructing coverings of R^n by translates of a given closed convex body K with as small a star number as possible. By a minor modification of method we used in [2] we prove

THEOREM 2. *Provided n is sufficiently large, if K is a closed convex body in R^n with difference body $\mathsf{D}K$, there is a covering of R^n by translates of K with star number less than*

$$\frac{V(\mathsf{D}K)}{V(K)}\{n\log n+n\log\log n+4n+1\}.$$

Here the ratio of the volumes $V(\mathsf{D}K)/V(K)$ is at most $\binom{2n}{n}$, in general, and is equal to 2^n if K is symmetric.

We can neither prove that general coverings by translates of a closed convex body must have a large star number, nor show that lattice coverings can be constructed with reasonably small star number. Indeed we can do neither of these two things in the special case of coverings by spheres.

2. Proof of Theorem 1. Suppose that the translates of the closed convex body K symmetrical in $\boldsymbol{o}$ by the vectors of a lattice Λ cover the whole of R^n. The points of the form $\frac{1}{2}\boldsymbol{a}$ with $\boldsymbol{a} \in \Lambda$ fall into 2^n congruence classes modulo the lattice Λ. Let $\boldsymbol{o}, \frac{1}{2}\boldsymbol{a}_1, \ldots, \frac{1}{2}\boldsymbol{a}_N$, with $\boldsymbol{a}_i \in \Lambda$ and $N = 2^n - 1$ be representatives of these congruence classes. Then as the translates of K by the vectors of Λ cover R^n, we can choose $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_N$ in Λ so that

$$\tfrac{1}{2}\boldsymbol{a}_i \in K + \boldsymbol{b}_i, \quad i = 1, 2, \ldots, N.$$

So

$$\tfrac{1}{2}\boldsymbol{a}_i - \boldsymbol{b}_i \in K, \quad i = 1, 2, \ldots, N.$$

As these points lie in different congruence classes modulo Λ and are not congruent to $\boldsymbol{o}$ modulo Λ, they are all different and are all different from $\boldsymbol{o}$. Indeed, as

$$-(\tfrac{1}{2}\boldsymbol{a}_i - \boldsymbol{b}_i) \equiv \tfrac{1}{2}\boldsymbol{a}_i - \boldsymbol{b}_i \pmod{\Lambda},$$

the $2N$ points

$$\pm(\tfrac{1}{2}\boldsymbol{a}_i - \boldsymbol{b}_i), \quad i = 1, 2, \ldots, N,$$

are all different and are in K, as K is symmetrical in $\boldsymbol{o}$. But, as

$$\pm(\tfrac{1}{2}\boldsymbol{a}_i - \boldsymbol{b}_i) \in K,$$

we have

$$\pm(\tfrac{1}{2}\boldsymbol{a}_i - \boldsymbol{b}_i) \in K \cap \{K \pm (\boldsymbol{a}_i - 2\boldsymbol{b}_i)\},$$

for $i = 1, 2, \ldots, N$. So K meets the $2^{n+1} - 1$ sets

$$K, \; K \pm (\boldsymbol{a}_i - 2\boldsymbol{b}_i), \quad i = 1, 2, \ldots, 2^n - 1,$$

of the covering. This proves the result.

3. Proof of Theorem 2. Let K be a closed convex body in R^n. After application of a suitable linear transformation we may assume that the volume of K is

$$V(K) = 8^{-n}.$$

Then by a result of Shephard and Rogers [3] the volume of the difference body $\mathsf{D}K$ satisfies

$$V(\mathsf{D}K) \leqslant \binom{2n}{n} V(K) < 2^{-n}.$$

So, by the Minkowski-Hlawka theorem, after application of a suitable linear transformation of determinant 1, we may suppose that the translates

$$\mathsf{D}K+\boldsymbol{g} \quad (\boldsymbol{g}\in\Lambda),$$

of $\mathsf{D}K$ by the vectors $\boldsymbol{g}$ of the lattice Λ of points with integral coordinates are disjoint.

Now take N to be the integer nearest to

$$8^n\{n\log n+n\log\log n+4n\},$$

and H to be the integer nearest to

$$V_0(1+\varepsilon)8^n\{n\log n+n\log\log n+4n\},$$

where

$$V_0=V(\mathsf{D}K), \quad \varepsilon=2^{-n/3}.$$

As in [2] we consider N points $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ in the cube C of points $\boldsymbol{x}=(x_1, \ldots, x_n)$ satisfying

$$0\leqslant x_i<1, \quad i=1,2,\ldots,n.$$

Let G_H be the set of points covered by H or more sets of the system

$$\mathsf{D}K+\boldsymbol{x}_i+\boldsymbol{g} \quad (i=1,\ldots,N,\ \boldsymbol{g}\in\Lambda);$$

and let $\delta(G_H)$ denote the volume of $G_H\cap C$. Then, as in [2], the mean value of $\delta(G_H)$ over all choices of $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ in C is

$$\mathscr{M}\big(\delta(G_H)\big)=\sum_{k=H}^{N}\frac{N!}{k!(N-k)!}V_0^k(1-V_0)^{N-k},$$

and

$$\log\mathscr{M}\big(\delta(G_H)\big)\leqslant(N-H)\log\left(1+\frac{H}{N-H}\right)-H\log\frac{H}{V_0N}+(N-H)\log(1-V_0)$$

$$-\frac{1}{2}\log\left(1-\frac{H}{N}\right)-\log\left(1-\frac{(N+1)V_0}{H+1}\right)+O(1)$$

$$\leqslant H-H\log\frac{H}{V_0N}-V_0N-\log\left(1-\frac{(N+1)V_0}{H+1}\right)+O(1).$$

But, by our choice of N and H,

$$H = V_0(1+\varepsilon)N+O(1), \quad (H+1) = V_0(1+\varepsilon)(N+1)+O(1),$$

so that

$$\log \mathscr{M}\big(\delta(G_H)\big) \leqslant NV_0[1+\varepsilon-(1+\varepsilon)\log(1+\varepsilon)-1]-\log[\varepsilon/(1+\varepsilon)]+O(1)$$
$$\leqslant -\tfrac{1}{2}\varepsilon^2 NV_0-\log\varepsilon+O(1).$$

Since

$$V_0 = V(\mathsf{D}K) \geqslant 2^n V(K) = 4^{-n},$$

we have

$$\varepsilon^2 NV_0 \geqslant 2^{-2n/3}8^n 4^{-n} = 2^{n/3},$$

and, certainly,

$$\log \mathscr{M}\big(\delta(G_H)\big) \leqslant -n\log n - n\log\log n - n\log 16-\log 2,$$

provided n is sufficiently large.

Also, as in [2], if n is sufficiently large and

$$\eta = 1/2n\log n,$$

the mean value of the density of the set E_0' of points belonging to no set of the system

$$(1-2\eta)K+\boldsymbol{x}_i+\boldsymbol{g} \quad (1 \leqslant i \leqslant N,\ \boldsymbol{g}\in\Lambda)$$

satisfies

$$\log \mathscr{M}\big(\delta(E_0')\big) \leqslant -n\log n - n\log\log n - n\log 16-\log 2.$$

So we can choose $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ in C so that

$$\delta(G_H)+\delta(E_0') < \left(\frac{1}{16n\log n}\right)^n = \eta^n V(K) < \eta^n V_0.$$

Just as in [2] it follows that the system of sets

(1) $$(1-\eta)K+\boldsymbol{x}_i+\boldsymbol{g} \quad (1 \leqslant i \leqslant N,\ \boldsymbol{g}\in\Lambda)$$

covers the whole of space, and that no point of space belongs to H of the sets

$$(1-\eta)\mathsf{D}K+\boldsymbol{x}_i+\boldsymbol{g} \quad (1 \leqslant i \leqslant N,\ \boldsymbol{g}\in\Lambda).$$

It follows immediately that the star number of the covering (1) is at most

$$H \leqslant \frac{V(\mathsf{D}K)}{V(K)}\{n\log n + n\log\log n + 4n+1\}.$$

We remark that it is easy to arrange, in addition, that the density of the covering is at most

$$n\log n + n\log\log n + 4n,$$

and that no point of space is covered more than

$$e(n\log n + n\log\log n + 4n)$$

times.

We also note that the covering of space that we have constructed by using the sets (1) has the property that, if $\boldsymbol{a}$ is any vector, then the set $(1-\eta)K+\boldsymbol{a}$ meets at most

$$H \leqslant \frac{V(\mathsf{D}K)}{V(K)}\{n\log n + n\log\log n + 4n + 1\}$$

of the sets of the system (1). This is nearly best possible, since a simple averaging argument shows, that if a system of convex sets

$$K + \boldsymbol{a}_i, \quad i = 1, 2, \ldots,$$

covers space and has density δ, then there is a point $\boldsymbol{a}$ such that $K+\boldsymbol{a}$ meets at least

$$\frac{V(\mathsf{D}K)}{V(K)}\,\delta$$

sets of the system.

References

[1] V. Boltyanskiĭ, *On a property of two-dimensional compacta*, Doklady Akad. Nauk SSSR 75 (1950), pp. 605-608.

[2] P. Erdös and C. A. Rogers. *Covering space with convex bodies*, Acta Arithmetica 7 (1962), pp. 281-285.

[3] C. A. Rogers and G. C. Shephard, *The difference body of a convex body*, Archiv der Mathematik 8 (1957), pp. 220-233.

Reçu par la Rédaction le 10. 4. 1963