A THEOREM ON UNIFORM DISTRIBUTION

by

H. DAVENPORT and P. ERDŐS

1. It is well known that if α is any positive irrational number, the sequence

(1) $$\alpha,\ 2\alpha,\ 3\alpha, \ldots$$

is uniformly distributed modulo 1. A more general concept of uniform distribution was introduced by LEVEQUE[1]. Let

$$z_1 < z_2 < \ldots$$

be a sequence of positive real numbers, such that $z_n \to \infty$ as $n \to \infty$. For given λ, with $0 < \lambda < 1$, let $F(N)$ denote the number of positive integers $k \leqq N$ for which $k\alpha$ falls in one of the intervals

(2) $$(z_j,\ z_j + \lambda(z_{j+1} - z_j)).$$

If $F(N)/N \to \lambda$ as $N \to \infty$, for each λ, we say that the sequence (1) is uniformly distributed relative to the sequence z_j. If $z_j = j$ we get the usual definition of uniform distribution modulo 1. The definition applies, of course, to sequences other than (1), but in the present paper we limit ourselves to this case. We shall suppose that

(3) $$z_{j+1}/z_j \to 1$$

as $j \to \infty$, since otherwise (as is easily seen) the sequence (1) cannot be uniformly distributed relative to $\{z_j\}$ for any α.

It follows from the work of LEVEQUE, supplemented by that of DAVENPORT and LEVEQUE[2] that *provided $z_{j+1}-z_j$ is monotonic (in the wide sense), the sequence* (1) *is uniformly distributed relative to $\{z_j\}$ for almost all $\alpha > 0$*, i.e. for almost all α in any interval (α_1, α_2), where $\alpha_2 > \alpha_1 > 0$.

We conjecture that this remains true without the requirement that $z_{j+1}-z_j$ should be monotonic. We are unable to prove this, but we shall prove that the requirement can be omitted provided that the numbers z_j are not very dense.

[1] LEVEQUE, W. J., "On uniform distribution modulo a subdivision", *Pacific J. of Math.*, **3** (1953), 757—771.

[2] DAVENPORT, H.—LEVEQUE, W. J., "Uniform distribution relative to a fixed sequence", *Michigan Math. J.* **10** (1963), 315—319.

It is convenient to consider a somewhat more general situation. Let

$$(x_1, y_1), (x_2, y_2), \ldots \tag{4}$$

be a sequence of non-overlapping intervals, arranged in increasing order, such that $x_j \to \infty$ as $j \to \infty$, and let $I(Z)$ denote the measure of the part of the interval $(0, Z)$ which is contained in these intervals. Thus

$$\begin{cases} I(Z) = (y_1 - x_1) + \ldots + (y_j - x_j) \text{ if } y_j \leqq Z \leqq x_{j+1}, \\ I(Z) = (y_1 - x_1) + \ldots + (y_{j-1} - x_{j-1}) + (Z - x_j) \text{ if } x_j \leqq Z \leqq y_j. \end{cases} \tag{5}$$

Let $F_\alpha(N)$ denote the number of positive integers $k \leqq N$ for which $k\alpha$ falls in one of the intervals (4). Then we may expect that, under suitable conditions, $F_\alpha(N)$ will be approximated by $\alpha^{-1} I(N\alpha)$ for almost all α. We prove the following

Theorem. *Suppose that*[3]

$$I(Z) \gg Z. \tag{6}$$

Let $X(N)$ denote the number of j for which $x_j \leqq N$, and suppose that

$$X(N) \ll N^{2-\delta} \tag{7}$$

for some fixed $\delta > 0$. Then

$$\alpha F_\alpha(N) / I(N\alpha) \to 1 \text{ as } N \to \infty \tag{8}$$

for almost all $\alpha > 0$.

If we take

$$x_j = z_j, \; y_j = z_j + \lambda(z_{j+1} - z_j), \tag{9}$$

where $0 < \lambda < 1$, then it follows from (3) that $I(Z)/Z \to \lambda$, and we deduce that *the sequence* (1) *is uniformly distributed relative to $\{z_j\}$ for almost all α, provided that the number of $z_j < N$ is $\ll N^{2-\delta}$.*

We conjecture that the theorem stated above holds without the condition (7). How far the condition (6) can be relaxed is doubtful[4]; we have not been able to disprove the possibility that the result may hold merely if $I(Z) \to \infty$ as $Z \to \infty$.

We take the opportunity of drawing attention to a problem connected with uniform distribution which was proposed by Khintchine[5] and seems to be still unsolved. Let S be a set in $(0,1)$ which is measurable in the sense of Lebesgue, with measure $m(S)$. Let $F_\alpha(N, S)$ denote the number of positive integers $k \leqq N$ for which the fractional part of $k\alpha$ falls in S. Is it true that

$$F_\alpha(N, S)/N \to m(S) \text{ as } N \to \infty \tag{10}$$

for almost all α in $(0, 1)$?

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[3] We use VINOGRADOV's symbols $\gg$ and $\ll$ to indicate an inequality containing an unspecified positive constant factor.

[4] In the theorem as it stands, the condition (6) can be relaxed to some extent if (7) is correspondingly strengthened.

[5] KHINTCHINE, A., "Ein Satz über Kettenbrüche, mit arithmetischen Anwendungen", *Math. Zeitschrift*, **18** (1923), 289—306 (303—306).

A more general conjecture would be the following. Let $f(x)$ be a bounded, measurable, and non-negative function, and put

$$I(Z) = \int_0^Z f(x)\,dx.$$

Suppose that $I(Z) \to \infty$ as $Z \to \infty$. Is it true that, for almost all $\alpha > 0$,

$$\left\{\sum_{k=1}^{N} f(k\,\alpha)\right\} / I(N\,\alpha) \to \alpha^{-1}$$

as $N \to \infty$? This would include (10) on taking $f(x)$ to be the characteristic function of S in (0, 1) and periodic with period 1. It would also include the conjecture stated above that the conclusion of our theorem may hold merely if $I(Z) \to \infty$. We can make no contribution to the proof or disproof of these conjectures: it seems[6] that the condition of boundedness of $f(x)$ in the above cannot be much relaxed even if $f(x)$ is assumed to be periodic.

2. We first prove that the conclusion of the theorem will follow if we establish the inequality

$$\int_{\alpha_1}^{\alpha_2} (F_\alpha(N) - \alpha^{-1} I(N\,\alpha))^2\, d\,\alpha \ll N^{2-\delta}. \tag{11}$$

The argument is on well known lines.

We can choose an increasing sequence $N_1, N_2, \ldots$ of positive integers so that

$$N_{r+1}/N_r \to 1 \tag{12}$$

and

$$\sum_r N_r^{-2} \int_{\alpha_1}^{\alpha_2} (F_\alpha(N_r) - \alpha^{-1} I(N_r\,\alpha))^2 \; d\,\alpha \text{ converges;}$$

for instance, we can take $N_r = [r^\gamma]$ with any fixed $\gamma > \delta^{-1}$. It follows from a well known general theorem[7] that

$$\sum_r N_r^{-2} (F_\alpha(N_r) - \alpha^{-1} I(N_r\,\alpha))^2$$

converges for almost all α in (α_1, α_2), and in particular that

$$N_r^{-1} \mid F_\alpha(N_r) - \alpha^{-1} I(N_r\,\alpha) \mid \to 0 \tag{13}$$

as $r \to \infty$, for almost all α in (α_1, α_2).

If $N_r \leqq N < N_{r+1}$, we have

$$F_\alpha(N_r) \leqq F_\alpha(N) \leqq F_\alpha(N_r) + (N - N_r),$$

$$I(N_r\,\alpha) \leqq I(N\,\alpha) \leqq I(N_r\,\alpha) + \alpha(N - N_r),$$

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[6] Compare Erdős, P., "On the strong law of large numbers", *Trans. American Math. Soc.*, **67** (1949), 51—56.

[7] See Weyl, H., "Über die Gleichverteilung von Zahlen mod. Eins", *Math. Annalen*, **77** (1916), 313—352, § 7.

whence

$$|F_\alpha(N) - \alpha^{-1} I(N\alpha)| \leqq |F_\alpha(N_r) - \alpha^{-1} I(N_r\alpha)| + (N - N_r).$$

In view of (12), it follows from (13) that

$$N^{-1}|F_\alpha(N) - \alpha^{-1} I(N\alpha)| \to 0 \text{ as } N \to \infty$$

for almost all α, and in view of (6), this in turn implies (8). Thus (8) holds for almost all α.

3. Define the function $\psi(t)$ of a real variable t by

$$\psi(t) = \begin{cases} t - [t] - \dfrac{1}{2} & \text{if } t \text{ is not an integer,} \\ 0 & \text{if } t \text{ is an integer.} \end{cases} \tag{14}$$

For given N and α, define $J = J(\alpha)$ by

$$x_J \leqq N\alpha < x_{J+1}, \tag{15}$$

and write

$$G_\alpha(N) = \sum_{j=1}^{J(\alpha)} (\psi(x_j/\alpha) - \psi(y_j/\alpha)). \tag{16}$$

We shall prove that (11) will follow from the inequality

$$\int_{\alpha_1}^{\alpha_2} (G_\alpha(N))^2 \, d\alpha \ll N^{2-\delta}. \tag{17}$$

For any particular j, the number of values of k for which

$$x_j < k\alpha < y_j$$

is

$$\left[\frac{y_j}{\alpha}\right] - \left[\frac{x_j}{\alpha}\right] = \frac{y_j - x_j}{\alpha} + \psi\left(\frac{x_j}{\alpha}\right) - \psi\left(\frac{y_j}{\alpha}\right),$$

provided neither x_j/α nor y_j/α is an integer. A similar expression holds, with y_j/α replaced by N, for the number of values of k satisfying $x_j < k\alpha < N\alpha$. It follows, on recalling the definition of $I(N\alpha)$ in (5), that

$$F_\alpha(N) = \alpha^{-1} I(N\alpha) + G_\alpha(N) + O(1) + O(\nu(\alpha, N)),$$

where $\nu(\alpha, N)$ denotes the number of $k \leq N$ for which $k\alpha$ coincides with one of the numbers x_j or y_j. Since $\nu(\alpha, N) = 0$ for all but a finite number of values of α, it follows from the above relation that (17) implies (11).

4. To simplify writing, we shall now take $\alpha_1 = 1$ and $\alpha_2 = 2$. We recall that $\psi(t)$, defined in (14), has the expansion

$$\psi(t) = -\frac{1}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin 2\pi mt. \tag{18}$$

Hence

$$\pi G_\alpha(N) = G'_\alpha(N) + G''_\alpha(N),$$

where

$$G'_\alpha(N) = \sum_{j=1}^{J(\alpha)} \sum_{m=1}^{M} \frac{1}{m}\left(\sin 2\pi m y_j/\alpha - \sin 2\pi m x_j/\alpha\right), \tag{19}$$

$$G''_\alpha(N) = \sum_{j=1}^{J(\alpha)} \sum_{m>M} \frac{1}{m}\left(\sin 2\pi m y_j/\alpha - \sin 2\pi m x_j/\alpha\right). \tag{20}$$

Thus

$$\int_1^2 (G_\alpha(N))^2\, d\alpha \ll \int_1^2 (G'_\alpha(N))^2\, d\alpha + \int_1^2 (G''_\alpha(N))^2\, d\alpha . \tag{21}$$

We now estimate, in a very simple manner, the second of the two integrals on the right. By partial summation, we have

$$\left| \sum_{m>M} \frac{1}{m} \sin 2\pi m t \right| \ll \min\left(1, \frac{1}{M\|t\|}\right),$$

where $\|t\|$ denotes the distance of t from the nearest integer. Hence

$$|G''_\alpha(N)| \ll \sum_{j=1}^{J(\alpha)} \min\left(1, \frac{1}{M\|x_j/\alpha\|}\right) + \sum_{j=1}^{J(\alpha)} \min\left(1, \frac{1}{M\|y_j/\alpha\|}\right).$$

It will suffice to treat the first of these sums. By Cauchy's inequality,

$$\left\{\sum_{j=1}^{J(\alpha)} \min\left(1, \frac{1}{M\|x_j/\alpha\|}\right)\right\}^2 \leqq J(\alpha) \sum_{j=1}^{J(\alpha)} \min\left(1, \frac{1}{M^2\|x_j/\alpha\|^2}\right).$$

By (15), we have $J(\alpha) \leqq X(N\alpha) \leqq X(2N)$. For each j,

$$\int_1^2 \min\left(1, \frac{1}{M^2\|x_j/\alpha\|^2}\right) d\alpha = x_j \int_{x_j/2}^{x_j} \min\left(1, \frac{1}{M^2\|\beta\|^2}\right) \frac{d\beta}{\beta^2},$$

on putting $\alpha = x_j/\beta$; and since

$$\int_\nu^{\nu+1} \min\left(1, \frac{1}{M^2\|\beta\|^2}\right) \frac{d\beta}{\beta^2} \ll \frac{1}{M\nu^2}$$

for any positive integer ν, it follows that

$$\int_1^2 \min\left(1, \frac{1}{M^2\|x_j/\alpha\|^2}\right) d\alpha \ll \frac{x_j}{M} \sum_{\nu > x_j/2} \frac{1}{\nu^2} \ll \frac{1}{M}.$$

Hence

$$\int_1^2 (G''_\alpha(N))^2\, d\alpha \ll M^{-1}(X(2N))^2 . \tag{22}$$

5. It remains to estimate the analogous integral with $G'_\alpha(N)$. By (19), this is

$$\int_1^2 \left\{ \sum_{j=1}^{J(\alpha)} \sum_{m=1}^{M} \frac{1}{m} (\sin 2\pi m y_j/\alpha - \sin 2\pi m x_j/\alpha) \right\}^2 d\alpha .$$

We write the square of the sum as a double sum, over j, k going from 1 to $J(\alpha)$ and over m, n going from 1 to M, and interchange summation and integration. The maximum value of $J(\alpha)$ is $X(2N)$, and for given j, k the conditions $j \leqq J(\alpha)$, $k \leqq J(\alpha)$ are equivalent to

$$\alpha \geqq \max(x_j/N, x_k/N) .$$

Hence the integral in question is

$$\sum_{j=1}^{X(2N)} \sum_{k=1}^{X(2N)} \sum_{m=1}^{M} \sum_{n=1}^{M} \frac{1}{mn} \int_{\alpha_{jk}}^{2} S(j, m)\, S(k, n)\, d\alpha ,$$

where

$$\alpha_{jk} = \max(1, x_j/N, x_k/N)$$

and

$$S(j, m) = \sin 2\pi m y_j/\alpha - \sin 2\pi m x_j/\alpha ,$$

$$S(k, n) = \sin 2\pi n y_k/\alpha - \sin 2\pi n x_k/\alpha .$$

Putting $\alpha = 1/\beta$, the expression becomes

$$\sum_{j=1}^{X(2N)} \sum_{k=1}^{X(2N)} \sum_{m=1}^{M} \sum_{n=1}^{M} \frac{1}{mn} \int_{\frac{1}{2}}^{\beta_{jk}} S(j, m)\, S(k, n)\, \beta^{-2}\, d\beta , \tag{23}$$

where

$$\beta_{jk} = \min(1, N/x_j, N/x_k)$$

and $S(j, m)$, $S(k, n)$ are defined as above with $1/\alpha$ replaced by β. We have

$$\begin{aligned} 2S(j, m)\, S(k, n) = {} & \cos 2\pi(my_j - ny_k)\beta - \cos 2\pi(my_j + ny_k)\beta \\ & - \cos 2\pi(mx_j - ny_k)\beta + \cos 2\pi(mx_j + ny_k)\beta \\ & - \cos 2\pi(my_j - nx_k)\beta + \cos 2\pi(my_j + nx_k)\beta \\ & + \cos 2\pi(mx_j - nx_k)\beta - \cos 2\pi(mx_j + nx_k)\beta . \end{aligned}$$

We arrange this as a sum of pairs, such as

$$\cos 2\pi(my_j - ny_k)\beta - \cos 2\pi(mx_j - ny_k)\beta .$$

It will suffice to consider this pair, the treatment of the other pairs being similar. Thus we have to estimate the expression

$$\sum_{j=1}^{X(2N)} \sum_{k=1}^{X(2N)} \sum_{m=1}^{M} \sum_{n=1}^{M} \frac{1}{mn} \int_{\frac{1}{2}}^{\beta_{jk}} \{\cos 2\pi\beta(my_j - ny_k) - \cos 2\pi\beta(mx_j - ny_k)\}\, \beta^{-2}\, d\beta . \tag{24}$$

We use two inequalities. The first is the obvious one:

$$\left|\int_{\frac{1}{2}}^{\beta_{jk}} (\cos 2\pi\beta A)\,\beta^{-2}\,da\right| \ll \min(1, |A|^{-1}), \tag{25}$$

where A is any real number and $\frac{1}{2} \leqq \beta_{jk} \leqq 1$. The second is

$$\left|\int_{\frac{1}{2}}^{\beta_{jk}} (\cos 2\pi\beta A - \cos 2\pi\beta B)\,\beta^{-2}\,d\beta\right| \ll |A - B| \min(1, |A + B|^{-1}), \tag{26}$$

valid provided $|A—B| \ll 1$. This follows from the fact that the integral on the left is

$$2 \int_{\frac{1}{2}}^{\beta_{jk}} \sin \pi\beta(B - A) \sin \pi\beta(B + A)\,\beta^{-2}\,d\beta =$$

$$= 2 \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)!} \pi^{2r+1} (B - A)^{2r+1} \int_{\frac{1}{2}}^{\beta_{jk}} \beta^{2r-1} \sin \pi\beta(A + B)\,d\beta,$$

together with the fact that

$$\left|\int_{\frac{1}{2}}^{\beta_{jk}} \beta^{2r-1} \sin \pi\beta(A + B)\,d\beta\right| < C \min(1, |A + B|^{-1}),$$

where C is independent of r.

We divide the sum (24) into two parts. For the terms with $m(y_j—x_j) \geqq 1$, the absolute value of the integral, by (25), is

$$\ll \min(1, |my_j - ny_k|^{-1}) + \min(1, |mx_j - ny_k|^{-1}).$$

For the terms with $m(y_j—x_j) < 1$, the absolute value of the integral, by (26), is

$$\ll m(y_j - x_j) \min(1, |mx'_j - ny_k|^{-1}),$$

where $x'_j = \frac{1}{2}(x_j + y_j)$. But here we can replace x'_j by x_j, since the term $|mx'_j—ny_k|^{-1}$ is significant only if $|mx'_j—ny_k| > 1$, and we have $m\,|x'_j—x_j| < \frac{1}{2}$. Thus we can put the two parts of the sum together again as

$$\sum_{j=1}^{X(2N)} \sum_{k=1}^{X(2N)} \sum_{m=1}^{M} \sum_{n=1}^{M} \frac{1}{mn} P(m, j)\, Q(m, n, j, k), \tag{27}$$

where

$$P(m, j) = \min(1, m(y_j - x_j)),$$

$$Q(m, n, j, k) = \min(1, |my_j - ny_k|^{-1}) + \min(1, |mx_j - ny_k|^{-1}).$$

It will be sufficient to deal with the first of the two terms in Q, the second being treated similarly.

For given k, m, n, consider first those values of j (if any) for which

$$ny_k - \frac{1}{2} \leqq my_j < ny_k + \frac{1}{2}.$$

Denote these by $j_1 \leqq j \leqq j_2$. We have

$$\sum_{j=j_1}^{j_2} P(m, j) \leqq 1 + \sum_{j=j_1+1}^{j_2} m(y_j - x_j).$$

For $j_1 + 1 \leqq j \leqq j_2$, the intervals (x_j, y_j) are disjoint and satisfy

$$x_j \geqq x_{j_1+1} > y_{j_1} \geqq \left(ny_k - \frac{1}{2}\right)/m\,,$$

$$y_j \leqq y_{j_2} < \left(ny_k + \frac{1}{2}\right)/m\,.$$

Hence

$$\sum_{j=j_1}^{j_2} P(m, j) \ll 1,$$

whence

$$\sum_{j=j_1}^{j_2} P(m, j)\, Q(m, n, j, k) \ll 1.$$

Consider next those values of j for which

$$ny_k + \nu - \frac{1}{2} \leqq my_j < ny_k + \nu + \frac{1}{2},$$

where ν is a non-zero integer. Denoting these by $j_1(\nu) \leqq j \leqq j_2(\nu)$ and arguing as before, we obtain

$$\sum_{j=j_1(\nu)}^{j_2(\nu)} P(m, j) \ll 1,$$

whence

$$\sum_{j=j_1(\nu)}^{j_2(\nu)} P(m, j)\, Q(m, n, j, k) \ll |\nu|^{-1}.$$

It follows that the multiple sum (27) has absolute value

$$\ll \sum_{k=1}^{X(2N)} \sum_{m=1}^{M} \sum_{n=1}^{M} \frac{1}{mn} \left(1 + \sum_{\nu} |\nu|^{-1}\right).$$

The greatest value of ν is $\ll MN$. Hence we obtain

$$\int_1^2 (G'_\alpha(N))^2\, d\alpha \ll (\log M)^2 (\log MN)\, X(2N). \tag{28}$$

6. We now take $M = [X(2N)]$. The hypothesis (7), together with (22), gives

$$\int_1^2 (G''_\alpha(N))^2\, d\alpha \ll N^{2-\delta},$$

and (28) gives

$$\int_1^2 (G'_\alpha(N))^2\, d\alpha \ll N^{2-\delta/2}.$$

As was shown in §§2 and 3, this suffices for the proof of the Theorem.

(Received June 21, 1963.)

ТЕОРЕМА О РАВНОМЕРНОМ РАСПРЕДЕЛЕНИИ

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Резюме

Авторы доказывают следующую теорему: Обозначим через $I(Z)$ меру общей части интервала $(0, Z)$ и суммарного множества интервалов $(x_1, y_1), \ldots, (x_n, y_n)$ не имеющих попарно общих частей $(0 \leqq x_1 < y_1 < x_2 < y_2 < \ldots)$. Предположим, что величина $I(z)/z$ ограничена снизу, если $z \geqq 1$. Обозначим через $X(N)$ число тех индексов j для которых $x_j \leqq N$. Предположим, что величина $X(N)/N^{2-\delta}$ ограничена сверху, где $\delta > 0$, $N = 1, 2, \ldots$. Пусть означает $F_\alpha(N)$ число тех чисел вида $\alpha, 2\alpha, \ldots, N\alpha$, которые попадают в один из интервалов (x_k, y_k). Тогда для почти всех значении α имеет место соотношение:

$$\lim_{N\to\infty} \frac{\alpha F_\alpha(N)}{I(\alpha N)} = 1 .$$