SOME REMARKS CONCERNING OUR PAPER „ON THE STRUCTURE OF SET-MAPPINGS". — NON-EXISTENCE OF A TWO-VALUED σ-MEASURE FOR THE FIRST UNCOUNTABLE INACCESSIBLE CARDINAL

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§ 1

In accordance with the notations of [4] we say that a cardinal m possesses property P_3 if every two-valued measure $\mu(X)$ defined on all subsets of a set S of power m vanishes identically, provided $\mu(\{x\})=0$ for every $x \in S$ and $\mu(X)$ is m-additive.

It was well known that $\aleph_0$ fails to possess property P_3 and that every cardinal $m<t_1$ possesses property P_3 where t_1 denotes the first uncountable inaccessible cardinal.

Recently A. TARSKI has proved, using a result of P. HANF, that a certain wide class of strongly inaccessible cardinals possesses property P_3 (called strongly incompact cardinals). H. J. KEISLER gave a purely set-theoretical proof of this result.[1] After having seen these papers we observed that the special case of this result that t_1 possesses property P_3 follows almost trivially from some of our theorems proved in [1]. We are going to give this simple proof in § 2. Our method for the proof is of purely combinatorial character, and although it is certainly weaker than that of A. TARSKI and H. J. KEISLER, we think that it is of interest to formulate how far one can go with these methods at present.

Let $t_0, \ldots, t_\xi, \ldots$ denote the increasing sequence of the strongly inaccessible cardinals ($t_0=\aleph_0$) and let Θ_ξ denote the initial number of t_ξ. We can prove similarly as in the case of t_1 that t_ξ possesses property P_3, provided $0<\xi<\Theta_\xi$. We only give the outline of this proof. Finally, we are going to formulate some problems.

§ 2

Let m, n be cardinal numbers. The partition symbol $m \to (n)^{<\aleph_0}$ introduced by P. ERDŐS and R. RADO in [5] denotes that the following statement is true:

Whenever S is a set, $\overline{\overline{S}}=m$, $[S]^k=I_1^k \cup I_2^k$ for every $k=1,2,\ldots$, then there exists a subset $S_0 \subseteqq S$ and a sequence $(\varepsilon_1, \ldots, \varepsilon_k, \ldots)$ ($\varepsilon_k=1$ or 2) such that $\overline{\overline{S}}_0=n$ and $[S_0]^k \subseteqq I_{\varepsilon_k}^k$ for every $k=1,2,\ldots$. $m \nrightarrow (n)^{<\aleph_0}$ denotes, as usual, the negation of this statement. ($[S]^k$ denotes the set $\{X: X \subseteqq S \wedge \overline{\overline{X}}=k\}$.)

In [1] we have proved the following theorems.

[1] See [2] and [3].

THEOREM 1. *If m does not possess property* P_3 *for a strongly inaccessible number* $m > \aleph_0$, *then*

$$m \to (m)^{<\aleph_0}$$

holds.

This is Theorem 9/a of [1].[2]

THEOREM 2. $m \nrightarrow (\aleph_0)^{<\aleph_0}$ *for every* $m < t_1$.

This is Theorem 9/b of [1].[3]

We did not observe that the following theorem follows almost trivially from Theorem 2:

THEOREM 3. $t_1 \nrightarrow (\aleph_1)^{<\aleph_0}$.

PROOF. Let S be a set, $\overline{\overline{S}} = t_1$. Let $S = \{x_\mu\}_{\mu < \Theta_1}$ be a well-ordering of type Θ_1 of S. Put $S_\mu = \{x_\varrho\}_{\varrho < \mu}$. Then $\overline{\overline{S}}_\mu < t_1$ for every $\mu < \Theta_1$. Thus by Theorem 2 for every $\mu < \Theta_1$ there exist sets $I^k_{1,\mu}, I^k_{2,\mu}$ satisfying the following conditions:

(1) $[S_\mu]^k = I^k_{1,\mu} \cup I^k_{2,\mu}$ for $k = 1, 2, \ldots$ and for every $\mu < \Theta_1$.

(2) If $X \subseteqq S_\mu$ and $\overline{\overline{X}} \geqq \aleph_0$, then there exists an integer $k_\mu > 0$ such that $[X]^{k_\mu} \subseteqq\!\!\!\!\!/\; I^{k_\mu}_{1,\mu}$ and $[X]^{k_\mu} \subseteqq\!\!\!\!\!/\; I^{k_\mu}_{2,\mu}$.

We are going to define the sets I^k_1, I^k_2 as follows:

(3) Put $I^1_1 = [S]^1$, $I^1_2 = 0$.

Let $X = \{x_{\mu_1}, \ldots, x_{\mu_k}, x_\mu\}$ $(\mu_1 < \ldots < \mu_k < \mu)$ be an arbitrary element of $[S]^{k+1}$ for $k = 1, 2, \ldots$.

Put

$$X \in I^{k+1}_1 \text{ if and only if } \{x_{\mu_1}, \ldots, x_{\mu_k}\} \in I^k_{1,\mu},$$
$$X \in I^{k+1}_2 \text{ if and only if } \{x_{\mu_1}, \ldots, x_{\mu_k}\} \in I^k_{2,\mu}. \tag{4}$$

It follows immediately from (1), (3) and (4) that we have

$$[S]^k = I^k_1 \cup I^k_2 \quad \text{for} \quad k = 1, 2, \ldots. \tag{5}$$

Suppose now that $S_0 \subseteqq S$, $\overline{\overline{S}}_0 = \aleph_1$. We prove:

(6) There exists an integer $k > 0$ such that $[S_0]^{k+1} \neq I^{k+1}_1$ and $[S_0]^{k+1} \subseteqq\!\!\!\!\!/\; I^{k+1}_2$.

In fact, S_0 contains a subset of type $\omega + 1$, i. e. there exists an $S_1 \subset S$ such that

$$S_1 = \{x_{\mu_0}, \ldots, x_{\mu_s}, \ldots, x_\mu\}_{s < \omega} \quad (\mu_0 < \ldots < \mu_s < \ldots < \mu)_{s < \omega}$$

[2] In fact, in [1] we stated the hypothesis (called hypothesis **) that m fails to possess property P_3 for every strongly inaccessible cardinal $m > \aleph_0$, but is obviuos that the proof given there makes use of this hypothesis only for the cardinal m in question.

[3] This theorem was first proved by G. FODOR.

Put $S_2=\{x_{\mu_0},\dots,x_{\mu_s},\dots\}_{s<\omega}$. Then $S_2 \subseteqq S_\mu$ and $\overline{\overline{S_2}}=\aleph_0$. Thus by (2) there exists an integer $k=k_\mu>0$ such that

$$[S_2]^k \nsubseteq I^k_{1,\mu} \quad \text{and} \quad [S_2]^k \nsubseteq I^k_{2,\mu}.$$

It follows from (4) that then $[S_1]^{k+1} \nsubseteq I_1^{k+1}$ and $[S_1]^{k+1} \nsubseteq I_2^{k+1}$, hence the same holds for S_0. (5) and (6) prove Theorem 3.

As an immediate consequence of Theorems 1 and 3 we get that

t_1 possesses property P_3.

As an immediate generalization of Theorem 2 with the methods used in [1] one can prove the following

THEOREM 4. *$m \nrightarrow (n)^{<\aleph_0}$, provided that m is strongly accessible from n and n is either $\aleph_0$ or is not strongly inaccessible.*[4]

Similarly to Theorem 3 it follows that we have

THEOREM 5. *Suppose n is either $\aleph_0$ or is not strongly inaccessible and let t_ξ denote the least strongly inaccessible cardinal greater than n. Then $t_\xi \nrightarrow (n^+)^{<\aleph_0}$.*

It is obvious that Theorem 1 and Theorem 5 imply that

t_ξ possesses property P_3, *provided $\xi<\Theta_\xi$.*

Let ξ_0 be the least ordinal number for which $\xi_0=\Theta_{\xi_0}$. We can not prove with our methods that t_{ξ_0} possesses property P_3 and we can prove the non-existence of a non-trivial σ-measure (with the well-known arguments) for t_ξ only if $\xi<\xi_0$.

§ 3. Problems

We say that the cardinal m possesses property Q if the solution of the so-called ramification problem is negative for it (see [4]).

Let us say that m possesses property S if $m \nrightarrow (m)^{<\aleph_0}$ holds.

It follows from Theorem 1 that property S is stronger than property P_3, provided $m>\aleph_0$ is strongly inaccessible.

(Note that contrary to the other properties investigated so far $m=\aleph_0$ possesses property S.)

The simplest unsolved problem concerning property S is

PROBLEM 1. $t_{\xi_0} \nrightarrow (t_{\xi_0})^{<\aleph_0}$?

We mention that we can not compare property Q with property S in the general case.

As to the symbol $m \rightarrow (n)^{<\aleph_0}$ it seems that the most interesting and most simple unsolved problem is

PROBLEM 2. $t_1 \rightarrow (\aleph_0)^{<\aleph_0}$?

[4] The proof of Theorem 4 as well as the proof of some other results concerning the symbol $m \rightarrow (n)^{<\aleph_0}$ will be published later. It is obvious that combining the ideas of Theorems 2 and 3 stronger negative results can be proved by transfinite induction. However, these results seem not to be best-possible and certainly do not help us in solving the problem P_3.

Without assuming that P_3 is false for an m we can not even prove the existence of an m for which

$$m \to (\aleph_0)^{<\aleph_0}$$

If we generalize in an obvious way the definition of the symbol $m \to (n)^{<\aleph_0}$ for order types α, β instead of cardinals m, n respectively, then we see at once that the proof of Theorem 3 gives the stronger result

$$\Theta_1 \not\to (\omega+1)^{<\aleph_0}.$$

The following problems remain open:

PROBLEM 3. $\Theta_1^+ \not\to (\omega+1)^{<\aleph_0}$? (or at least $\Theta_1^+ \not\to (\omega+2)^{<\aleph_0}$?).

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References

[1] P. ERDŐS and A. HAJNAL, On the structure of set-mappings, *Acta Math. Acad. Sci. Hung.*, **9** (1958), pp. 111–131.

[2] A. TARSKI, Some problems and results relevant to the foundations of set theory, *Proceedings of the International Congress for Logic, Methodology and Philosophy of Science* (Stanford, 1960).

[3] H. J. KEISLER, Some applications of the theory of models to set theory, *Proceedings of the International Congress for Logic, Methodology and Philosophy of Science* (Stanford, 1960).

[4] P. ERDŐS and A. TARSKI, On some problems involving inaccessible cardinals, *Esseys on the foundations of mathematics. Magnes Press. The Hebrew University of Jerusalem* (1961), pp. 50–82.

[5] P. ERDŐS and R. RADO, A partition calculus in set theory, *Bull. Amer. Math. Soc.*, **62** (1956), pp. 427–489.