PARTITION RELATIONS CONNECTED WITH THE CHROMATIC NUMBER OF GRAPHS

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1. The chromatic number of a combinatorial graph Γ is the least cardinal number a which has the following property. The set of nodes of Γ can be divided into a subsets in such a way that no edge of Γ joins two nodes belonging to the same subset. The simplest example of a graph of chromatic number a is the complete graph of order a, which has exactly a nodes each two of which are joined by an edge. A tree, *i.e.* a graph without circuits, has a chromatic number which is at most equal to two. More generally, this holds for every even graph, *i.e.* a graph all of whose circuits have an even number of edges. It is known‡ [**1**] that there are finite graphs without triangles whose chromatic number has any prescribed finite value a (Theorem 1). The construction used in [**1**] fails when a is infinite. The first part of this paper is concerned with a construction, modelled on that of [**1**] but differing from it in some essential respects, which yields a graph Γ_a, without triangles, of any given chromatic number $a \geqslant \aleph_0$ (Theorem 2). Under the assumption of a form of the general continuum hypothesis the set of nodes of such a graph can be made as small as it can be, *i.e.* of cardinal a (Theorem 3).

In the second part a new type of set-theoretical partition relation will be introduced, formed in analogy to partition relations studied in [**2**], which refers to a generalization of the notion of the Baire categories in analysis. For this relation we prove a result (Theorem 4) which might be considered as a wide generalization of a special case of a theorem of Dushnik and Miller§. It is worth noting that the last named theorem holds for any infinite value of the cardinal number a entering in its statement whereas Theorem 4 will only be proved for every regular infinite a. By means of Theorem 2 we shall in fact prove (Theorem 5) that the conclusion of Theorem 4 is false for every singular infinite cardinal, under the assumption of a form of the general continuum hypothesis.

2. Set union, difference, intersection and inclusion in the wide sense, are denoted by $A+B$, $A-B$, AB, $A \subset B$ respectively, and $A-B$ is used irrespective whether $B \subset A$ is true or false. The set of all mappings of B into A is A^B. The cardinal (number) of A is $|A|$, and the cardinal of an ordinal (number) n is $|n|$. Occasionally we shall use the obliteration operator ^ whose effect is to remove from a well-ordered series the term

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‡ In fact, the graph constructed in [**1**] has no triangle, no quadrilateral and no pentagon. In the present note quadrilaterals and pentagons will not be excluded.

§ [**3**], also [**2**], Theorem 44.

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above which it is placed. Thus $\{x_0, x_1, \ldots, \hat{x}_n\}$ (n finite) means just $\{x_0, x_1, \ldots, x_{n-1}\}$, whether or not the x's are distinct. This operator may even be placed above a symbol which has not yet been defined. If m and n are ordinals and $m \leqslant n$ then $[m, n)$ denotes the set of all ordinals ν such that $m \leqslant \nu < n$. Brackets $\{\}$ are used exclusively in order to specify a set by giving a list of its elements, and (x, y) denotes an ordered pair. Thus $[m, n) = \{\nu\colon m \leqslant \nu < n\}$. The next larger cardinal to a is denoted by a^+. For any cardinal $a \geqslant 2$ we denote by a' the least cardinal b such that, for some index set N satisfying $|N| = b$ and suitable cardinals $a_\nu < a$, we have $a = \Sigma(\nu \varepsilon N)\, a_\nu$; the cardinal a is regular if $a' = a$, and singular if $a' < a$.

For any set A the symbol $[A]^2$ denotes the set whose elements are all subsets $\{x, y\}_{\neq}$ of A of cardinal 2. A graph is a pair $\Gamma = (S, T)$ of sets such that $T \subset [S]^2$. The order $\phi(\Gamma)$ of Γ is defined by $\phi(\Gamma) = |S|$, and the chromatic number $\chi(\Gamma)$ is the least cardinal a such that, for some index set N of cardinal a, there is a partition $S = \Sigma(\nu \varepsilon N)\, S_\nu$ such that $[S_\nu]^2\, T = \emptyset$ for all $\nu \varepsilon N$.

Clearly, $\chi(\Gamma) \leqslant \phi(\Gamma)$. If Γ is complete, *i.e.* $T = [S]^2$, then $\chi(\Gamma) = \phi(\Gamma)$. The result of [**1**], as far as triangles are concerned, states that, given any finite cardinal a, there is a finite graph Γ_a such that $\chi(\Gamma_a) = a$ and, at the same time, $[\{x, y, z\}]^2 \not\subset T$ whenever $\{x, y, z\}_{\neq} \subset S$. In order to make it easier to follow our extension of this result to $a \geqslant \aleph_0$ we give a slightly modified version of the original proof of Kelly and Kelly for finite a.

THEOREM 1. *Corresponding to every $a < \aleph_0$ there exists a graph Γ_a, without triangles, such that $\phi(\Gamma_a) < \aleph_0$ and $\chi(\Gamma_a) = a$.*

Proof. It suffices to define an operator M which turns every graph Γ into a graph $M\Gamma$ such that

(i) $\phi(M\Gamma) = \phi(\Gamma)\chi(\Gamma) + \phi(\Gamma)^{\chi(\Gamma)}$.

(ii) If $\chi(\Gamma) < \aleph_0$, then $\chi(M\Gamma) = \chi(\Gamma) + 1$.

(iii) If Γ does not contain any triangle then $M\Gamma$ does not contain any triangle.

For if such an operator has been found then the assertion of the theorem holds for the graph $\Gamma_a = M^a\,\Gamma_0$ obtained by a-fold iteration of M applied to the graph $\Gamma_0 = (\emptyset, \emptyset)$. Let $\Gamma = (S, T)$ be a graph, and let n be the initial ordinal belonging to the cardinal $\chi(\Gamma)$. We put $M\Gamma = \Gamma' = (S', T')$ where

$$S' = \{(\nu, x)\colon\ \nu < n;\ x \varepsilon S\} + \{(n, x_0, x_1, \ldots, \hat{x}_n)\colon\ x_0, \ldots, \hat{x}_n \varepsilon S\};$$

$$T' = \Big\{\{(\nu, x), (\nu, y)\}\colon\ \nu < n;\ \{x, y\} \varepsilon T\Big\} + \Big\{\{(\nu, x_\nu), (n, x_0, \ldots, \hat{x}_n)\}\colon\ \nu < n;\ x_0, \ldots, \hat{x}_n \varepsilon S\Big\}.$$

Then $T' \subset [S']^2$, and (i) and (iii) hold. By definition of n there is $f \varepsilon [0, n)^S$ such that $\{x, y\} \varepsilon T$ implies $f(x) \neq f(y)$. Define $f' \varepsilon [0, n+1)^{S'}$ by putting

$$f'\Big((\nu, x)\Big) = f(x) \quad (\nu < n;\ x \varepsilon S),$$

$$f'\Big((n, x_0, \ldots, \hat{x}_n)\Big) = n \quad (x_0, \ldots, \hat{x}_n \varepsilon S).$$

Then $\{\xi, \eta\} \varepsilon T'$ implies $f'(\xi) \neq f'(\eta)$, so that $\chi(\Gamma') \leqslant |n+1|$. If we now suppose that $\chi(\Gamma') = |n| < \aleph_0$, then there is $g' \varepsilon [0, n)^{S'}$ such that $\{\xi, \eta\} \varepsilon T'$ implies $g'(\xi) \neq g'(\eta)$. Define $g_\nu \varepsilon [0, n)^S$ by putting

$$g_\nu(x) = g'\Big((\nu, x)\Big) \quad (\nu < n;\ x \varepsilon S).$$

Let $\nu < n$; $\{x, y\} \varepsilon T$. Then $\{(\nu, x), (\nu, y)\} \varepsilon T'$;

$$g_\nu(x) = g'\Big((\nu, x)\Big) \neq g'\Big((\nu, y)\Big) = g_\nu(y).$$

By definition of n, and since n is finite, there is $x_\nu \varepsilon S$ such that $g_\nu(x_\nu) = \nu$. Put $\nu_0 = g'\Big((n, x_0, \ldots, \hat{x}_n)\Big)$. Then

$$\nu_0 < n;\ \{(\nu_0, x_{\nu_0}),\ (n, x_0, \ldots, \hat{x}_n)\} \varepsilon T;$$

$$g'\Big((\nu_0, x_{\nu_0})\Big) = g_{\nu_0}(x_{\nu_0}) = \nu_0 = g'\Big((n, x_0, \ldots, \hat{x}_n)\Big)$$

which contradicts the definition of g'. Hence $\chi(\Gamma') = |n+1|$, and (ii) follows. This proves Theorem 1.

Clearly, this argument fails for $a \geqslant \aleph_0$ since in this case the existence of x_ν can no longer be inferred. All we know is that $|\{g_\nu(x) : x \varepsilon S\}| = |n|$ which does not imply that $g_\nu(x)$ takes every value in $[0, n)$.

4. THEOREM 2. *Corresponding to every cardinal* $a \geqslant \aleph_0$ *there exists a graph* Γ_a *which has the following properties*:

(i) Γ_a *does not contain any triangle.*

(ii) $\chi(\Gamma_a) = a'$; $\phi(\Gamma_a) \geqslant a$.

(iii) *If* $a_0 < a$ *implies* $2^{a_0} \leqslant a$, *then* $\phi(\Gamma_a) = a$.

THEOREM 3. *Let* $a \geqslant \aleph_0$. *Then there exists a graph* Γ_a', *without triangles, such that* $\chi(\Gamma_a') = a$. *If*

$$a = \sup (b \varepsilon B)\, b', \tag{1}$$

for some non-empty set B *of infinite cardinals such that* $b_0 < b \varepsilon B$ *implies* $2^{b_0} \leqslant b$, *then* Γ_a' *can be made to satisfy, in addition,* $\phi(\Gamma_a') = a$. *Such a set* B *exists, for instance, when either* (i) a *is regular, and* $a_0 < a$ *implies* $2^{a_0} \leqslant a$, *or* (ii) a *is singular, and* $\aleph_0 \leqslant a_0 < a$ *implies* $2^{a_0} = a_0^+$.

5. *Proof of Theorem* 2. Let $a \geqslant \aleph_0$, and denote by m and n the initial ordinals belonging to a' and a respectively. We define sets $S_{a\nu}$, $T_{a\nu}$ for

$\nu < n$ as follows. Let $\nu_0 < n$, and suppose that $S_{a\nu}$ and $T_{a\nu}$ have been defined for $\nu < \nu_0$. Then we let $S_{a\nu_0}$ be the set of all pairs (ν_0, A) such that

$$A \subset \Sigma(\nu < \nu_0)\, S_{a\nu}; \quad |A| < a'; \quad [A]^2 \Sigma(\nu < \nu_0)\, T_{a\nu} = \varnothing.$$

In particular, $(\nu_0, \phi) \in S_{a\nu_0}$, so that $S_{a\nu_0} \neq \varnothing$.

Let $T_{a\nu_0}$ be the set of all sets $\{x, (\nu_0, A)\}_{\neq}$ such that $(\nu_0, A) \in S_{a\nu_0}$; $x \in A$. This completes the definition of $S_{a\nu}$, $T_{a\nu}$ for $\nu < n$ and it follows that

$$S_{a\mu} S_{a\nu} = \varnothing \quad (\mu < \nu < n).$$

Put $\quad S_a = \Sigma(\nu < n)\, S_{a\nu}; \quad T_a = \Sigma(\nu < n)\, T_{a\nu}; \quad \Gamma_a = (S_a, T_a).$

Then $$|S_a| = \Sigma(\nu < n)\, |S_{a\nu}| \geqslant \Sigma(\nu < n)\, 1 = a. \qquad (2)$$

Also $\quad T_a = \Sigma(\nu < n) \left\{ \{x, (\nu, A)\}_{\neq} : (\nu, A) \in S_{a\nu};\ x \in A \right\} \subset [S_a]^2$

so that Γ_a is a graph. In the remainder of the proof of Theorem 2 we shall suppress the suffix a.

Proof of (i). Let $[\{x_0, x_1, x_2\}_{\neq}]^2 \subset T$. We have to deduce a contradiction. We may assume that

$$x_\alpha = (\nu_\alpha, A_\alpha) \in S_{\nu_\alpha} \quad (\alpha < 3); \quad \nu_0 < \nu_1 < \nu_2 < n.$$

Let $\alpha < \beta < 3$. Then

$$\{x_\alpha, (\nu_\beta, A_\beta)\} = \{x_\beta, (\nu_\alpha, A_\alpha)\} = \{x_\alpha, x_\beta\} \in T$$

and therefore either $x_\alpha \in A_\beta$ or $x_\beta \in A_\alpha$. Now

$$\{x_\beta\} A_\alpha \subset S_{\nu_\beta} \Sigma(\nu < \nu_\alpha)\, S_\nu = \varnothing$$

and hence $$x_\alpha \in S_{\nu_\alpha} A_\beta \subset S_{\nu_\alpha} \Sigma(\nu < \nu_\beta)\, S_\nu; \ \nu_\alpha < \nu_\beta;$$

$$\{x_\alpha, x_\beta\} = \{x_\alpha, (\nu_\beta, A_\beta)\} \in T_{\nu_\beta}.$$

Therefore $$\{x_0, x_1\} \in [A_2]^2 T_{\nu_1} \subset [A_2]^2 \Sigma(\nu < \nu_2)\, T_\nu = \varnothing,$$

by definition of A_2. This is the desired contradiction, and (i) follows.

Proof of (ii). Define $f \in [0, m)^S$ as follows. Well-order S in such a way that whenever $\mu < \nu < n$; $x \in S_\mu$; $y \in S_\nu$, then $x < y$. Let $x_0 \in S$, and suppose that $f(x)$ has been defined for $x < x_0$. Then $x_0 = (\nu_0, A_0) \in S_{\nu_0}$, for some $\nu_0 < n$ and some $A_0 \subset \Sigma(\nu < \nu_0)\, S_\nu$, and $f(x)$ has already been defined for $x \in A_0$. Also, $|A_0| < a' = |m|$, so that there exists an ordinal $f(x_0) < m$ such that $f(x_0) \neq f(x)$ $(x \in A_0)$. This defines $f(x)$ for $x \in S$. Now let $\{y, x\} \in T$. We want to prove $f(y) \neq f(x)$. We may assume that $x = (\nu, A) \in S_\nu$; $y \in A$. Then by definition of $f(x)$, we have $f(x) \neq f(y)$. This shows that $f(x)$ is an admissible " colouring " of Γ with $|m|$ colours, so that $\chi(\Gamma) \leqslant |m| = a'$.

We shall now assume that

$$\chi(\Gamma) < a' \tag{3}$$

and derive a contradiction. Let k be the initial ordinal belonging to $\chi(\Gamma)$. Then there is $g \varepsilon [0, k)^S$ such that $g(x) \neq g(y)$ whenever $\{x, y\} \varepsilon T$. We define, for $\mu < m$, sets L_μ and ordinals ρ_μ as follows. Let $\mu_0 < m$, and suppose that L_μ and ρ_μ have been defined for $\mu < \mu_0$ and that

$$L_\mu \subset S; \quad \rho_\mu < n \quad (\mu < \mu_0).$$

Then, by Zorn's Lemma, there is a maximal set L_{μ_0} such that

$$L_{\mu_0} \subset S; \quad [L_{\mu_0}]^2 T = \emptyset; \quad g(x) \neq g(y) \text{ whenever } \{x, y\} \neq L_{\mu_0};$$
$$L_{\mu_0} \subset \Sigma\, (\rho_\mu < \nu < n)\, S_\nu, \text{ for each } \mu < \mu_0.$$

Then, by definition of a', $L_{\mu_0} \neq \emptyset$. Also,

$$|L_{\mu_0}| = |\{g(x) : x \varepsilon L_{\mu_0}\}| \leqslant |k| < a',$$

and it follows that there is an ordinal $\rho_{\mu_0} < n$ such that $L_{\mu_0} \subset \Sigma\, (\mu < \rho_{\mu_0})\, S_\mu$. This defines L_μ and ρ_μ for $\mu < m$. Put $\xi_\mu = (\rho_\mu, L_\mu)$ $(\mu < m)$. Then $\xi_\mu \varepsilon S_{\rho_\mu}$ $(\mu < m)$. Let $\mu_1 < \mu_0 < m$. Then

$$\emptyset \neq L_{\mu_0} \subset \left(\Sigma\, (\rho_{\mu_1} < \nu < n)\, S_\nu\right)\left(\Sigma (\nu < \rho_{\mu_0})\, S_\nu\right).$$

Hence there is ν such that $\rho_{\mu_1} < \nu < \rho_{\mu_0}$, so that $\rho_{\mu_1} < \rho_{\mu_0}$ $(\mu_1 < \mu_0 < m)$. Since $g(\xi_\mu) < k$ $(\mu < m)$, and $|k| < |m|$, there are ordinals α, β such that $\alpha < \beta < m;\ g(\xi_\alpha) = g(\xi_\beta)$. Put $L_\alpha' = L_\alpha + \{\xi_\beta\}$. Then

$$\xi_\beta = (\rho_\beta, L_\beta)\, \varepsilon\, S_{\rho_\beta} \subset \Sigma\, (\rho_\mu < \nu < n)\, S_\nu \quad (\mu < \alpha),$$

and hence, by definition of L_α,

$$L_\alpha' \subset \Sigma\, (\rho_\mu < \nu < n)\, S_\nu \quad (\mu < \alpha). \tag{4}$$

If we assume that there is $x \varepsilon L_\alpha$ such that

$$\{x, \xi_\beta\}\, \varepsilon\, T, \tag{5}$$

then $x \varepsilon L_\alpha \subset \Sigma\, (\nu < \rho_\alpha)\, S_\nu$; $x = (\nu_1, A)$, for some $\nu_1 < \rho_\alpha$;

$$\{x, (\rho_\beta, L_\beta)\} = \{\xi_\beta, (\nu_1, A)\} = \{x, \xi_\beta\}\, \varepsilon\, T,$$

and we have either $x \varepsilon L_\beta$ or $\xi_\beta \varepsilon A$. Now

$$\{x\} L_\beta \subset S_{\nu_1} \Sigma\, (\rho_\alpha < \nu < n)\, S_\nu = \emptyset,$$

so that, in view of $\rho_\beta > \rho_\alpha > \nu_1$,

$$\xi_\beta \varepsilon S_{\rho_\beta} A \subset S_{\rho_\beta} \Sigma\, (\nu < \nu_1)\, S_\nu = \emptyset.$$

This contradiction proves that (5) is false. We infer from the definition of L_α that

$$[L_\alpha']^2\, T = \emptyset. \tag{6}$$

If $x \varepsilon L_\alpha$, then $\{x, \xi_\alpha\} = \{x, (\rho_\alpha, L_\alpha)\} \varepsilon T_{\rho_\alpha} \subset T$; $g(x) \neq g(\xi_\alpha) = g(\xi_\beta)$. This implies, by definition of L_α, that

$$g(x) \neq g(y), \text{ if } \{x, y\}_{\neq} \subset L_\alpha'. \tag{7}$$

Finally, if $\xi_\beta \varepsilon L_\alpha$, then the contradiction

$$\xi_\beta \varepsilon S_{\rho_\beta} L_\alpha \subset S_{\rho_\beta} \Sigma (\nu < \rho_\alpha) S_\nu = \emptyset$$

follows. Hence $\xi_\beta \notin L_\alpha$, so that

$$L_\alpha \subset \neq L_\alpha'. \tag{8}$$

The set of relations (4), (6), (7), (8) constitutes a contradiction to the maximum property of L_α. Hence the assumption (3) was false and (ii) is established.

Proof of (iii). We suppose that a is such that $a_0 < a$ implies $2^{a_0} \leqslant a$. We begin by deducing that, whenever $b < a$, then $a^b \leqslant a$. If, first of all a is a limit number then, by [**4**],

$$a^b = \Sigma (a_0 < a) a_0{}^b \leqslant \Sigma (a_0 < a) 2^{a_0 b} \leqslant \Sigma (a_0 < a) a = a.$$

If, on the other hand, $a = c^+$ then

$$a^b \leqslant (2^c)^b = 2^{cb} \leqslant a.$$

We can now prove that $|S_\nu| \leqslant a$ $(\nu < n)$. Let $\nu_0 < n$, and suppose that $|S_\nu| \leqslant a$ for $\nu < \nu_0$. Then it follows from the definition of S_{ν_0} that

$$|S_{\nu_0}| \leqslant \Sigma (b < a') \Big(\Sigma (\nu < \nu_0) |S_\nu|\Big)^b \leqslant \Sigma (b < a')(a|\nu_0|)^b$$
$$\leqslant \Sigma (b < a') a^b \leqslant aa' = a.$$

This proves that $|S_\nu| \leqslant a$ $(\nu < n)$ and hence, by (2), that

$$a \leqslant |S| = \Sigma (\nu < n) |S_\nu| \leqslant a|n| = a,$$

and (iii) follows. This completes the proof of Theorem 2.

6. *Proof of Theorem* 3. If $a' = a$ then we may put $\Gamma_a' = \Gamma_a$. Now let $a' < a$, and let m be the initial ordinal of cardinal a'. Then $a = \Sigma (\mu < m) a_\mu$, for some suitable cardinals $a_\mu < a$. Let $\Gamma_a' = (S_a', T_a')$, where $S_a' = \{(\mu, x) : \mu < m;\ x \varepsilon S_{c_\mu}\}$,

$$T_a' = \Big\{\{(\mu, x), (\mu, y)\} : \mu < m;\ \{x, y\} \varepsilon T_{c_\mu}\Big\};\ c_\mu = a_\mu{}^+,$$

and S_{c_μ} and T_{c_μ} are the sets of nodes and edges respectively of the graph Γ_{c_μ} defined above. By Theorem 2

$$\chi(\Gamma_{c_\mu}) = c_\mu' = c_\mu$$

and therefore, by definition of Γ_a',

$$\chi(\Gamma_a') = \sup (\mu < m) c_\mu = a.$$

Let us now suppose that a satisfies (1) for some set B possessing the property given in Theorem 3. Then we modify our definition of $\Gamma_a{}'$ by putting $\Gamma_a{}' = (S_a{}', T_a{}')$, where $S_a{}' = \{(b, x): b \varepsilon B;\ x \varepsilon S_b\}$,

$$T_a{}' = \Big\{\{(b, x), (b, y)\}: b \varepsilon B;\ \{x, y\} \varepsilon T_b\Big\}.$$

We have $\chi(\Gamma_a{}') = \sup (b \varepsilon B) \chi(\Gamma_b) = \sup (b \varepsilon B) b' = a$ and, by Theorem 2 (iii),

$$a \leqslant \phi(\Gamma_a{}') \leqslant \Sigma\,(b \varepsilon B)\,\phi(\Gamma_b) = \Sigma\,(b \varepsilon B)\,b \leqslant a\,|B| = a.$$

Finally, if a satisfies (i) of Theorem 3 then the set $\{a\}$ can be used as B, and if a satisfies (ii) of Theorem 3 then the set $\{b;\ \aleph_0 \leqslant b < a\}$ can be used as B. This proves Theorem 3.

7. Our next theorems are most conveniently expressed in terms of a partition relation of the form

$$A \rightarrow (b, \Lambda)^2. \tag{9}$$

Here A is a set, b a cardinal number and Λ a set of sets. The relation (9) expresses, by definition, the proposition that, whenever $[A]^2 = K_0 + K_1$, there is $X \subset A$ such that

$$\text{either } [X]^2 \subset K_0;\ |X| = b$$

$$\text{or } \quad [X]^2 \subset K_1;\ X \varepsilon \Lambda.$$

The negation of (9) is denoted by

$$A \not\rightarrow (b, \Lambda)^2.$$

Let Ω be a set of sets. A set A is said to be of *first Ω-category* if there is $\Omega' \subset \Omega$ such that $|\Omega'| < |\Omega|$ and $A \subset \Sigma\,(X \varepsilon \Omega')\,X$, and otherwise of *second Ω-category*.

THEOREM 4. *Let Ω be a set of sets and suppose that $|\Omega|$ is a regular infinite cardinal. Let A be a set which is of second Ω-category, and denote by Λ_2 the set of all subsets of A which are of second Ω-category. Then*

$$A \rightarrow (\aleph_0, \Lambda_2)^2.$$

Remark 1. Let Ω be the set of all closed, nowhere dense sets of real numbers. Assume that $2^{\aleph_0} = \aleph_1$. Then a set A of real numbers is of econd Ω-category if, and only if, A is of second Baire category. For the complement of every closed set is the union of open intervals with rational endpoints, so that $|\Omega| = 2^{\aleph_0} = \aleph_1$. Now Theorem 4 shows that *if the nodes of a graph Γ, which does not contain any infinite complete subgraph, form a set A of real numbers of second Baire category then there is a subset X of A, of second Baire category, which is independent, i.e.* which is such that no two elements of X are joined by an edge of Γ (assuming $2^{\aleph_0} = \aleph_1$).

In the case of graphs of a more special type similar results have been obtained by F. Bagemihl [5] which are, however, not implied by our result.

Remark 2. If n is an infinite ordinal such that $|n|$ is regular then we may put, in Theorem 4,

$$\Omega = \Big\{\{\nu\}:\ \nu < n\Big\}\ ;\ A = [0,\ n).$$

A subset X of A is of second Ω-category if, and only if, $|X| = |n|$. Hence Theorem 4 states in this case that, in the notation of [**2**], $a \to (\aleph_0,\ a)^2$ *whenever* $a = a' \geqslant \aleph_0$. This is the theorem of Dusknik and Miller [**3**] in the special case of regular cardinals.

Proof of Theorem 4. We may assume that $\Omega = \{A_\nu:\ \nu < n\}$, and that n is an initial ordinal of cardinal $|\Omega|$ $(\geqslant \aleph_0)$. Let $[A]^2 = K_0 + K_1$. We have to find a subset X of A such that either

$$[X]^2 \subset K_0\ ;\ |X| = \aleph_0 \tag{10}$$

or

$$[X]^2 \subset K_1\ ;\ X \varepsilon \Lambda_2. \tag{11}$$

If $A \not\subset \Sigma\,(\nu < n)\,A_\nu$ then (11) holds for $X = \{\xi\}$, where ξ is any element of $A - \Sigma\,(\nu < n)\,A_\nu$. Now let $A \subset \Sigma\,(\nu < n)\,A_\nu$. For $x \,\varepsilon\, A$ we put

$$U_0(x) = \Big\{y:\ \{x,\ y\}\,\varepsilon\, K_0\Big\}.$$

Case 1. There are elements $x_0,\ \ldots,\ \hat{x}_{\omega_0}$ of A such that

$$x_k \,\varepsilon\, A\,\Pi\,(\lambda < k)\,U_0(x_\lambda)\,\varepsilon\,\Lambda_2 \quad (k < \omega_0).$$

Then (10) holds for $X = \{x_0,\ \ldots,\ \hat{x}_{\omega_0}\}$.

Case 2. There are $k,\ x_0,\ \ldots,\ \hat{x}_k$ such that $k < \omega_0$; $x_0,\ \ldots,\ \hat{x}_k \,\varepsilon\, A$ and, if

$$D = A\,\Pi\,(\lambda < k)\,U_0(x_\lambda),$$

then

$$D\,\varepsilon\,\Lambda_2\ ;\ DU_0(x) \,\bar{\varepsilon}\, \Lambda_2 \quad (x \,\varepsilon\, D).$$

Then we define $y_0,\ \ldots,\ \hat{y}_n$ as follows.

Let $\nu_0 < n$ and $y_0,\ \ldots,\ \hat{y}_{\nu_0} \,\varepsilon\, D$. If $D \subset \Sigma\,(\nu < \nu_0)\Big(\{y_\nu\} + U_0(y_\nu) + A_\nu\Big)$ then there are $\mu_0,\ \ldots,\ \hat{\mu}_{\nu_0} < n$ such that $D \subset \Sigma\,(\nu < \nu_0)\,\Sigma\,(\mu < \mu_\nu)\,A_\mu$. Now, since $|\nu_0| < |n| = |n|'$, we have $\overline{\mu} = \sup\,(\nu < \nu_0)\,\mu_\nu < n$ and therefore

$$D \subset \Sigma\,(\mu < \overline{\mu})\,A_\mu\ ;\ D \,\bar{\varepsilon}\, \Lambda_2,$$

which is a contradiction. Hence we can choose

$$y_{\nu_0} \,\varepsilon\, D - \Sigma\,(\nu < \nu_0)\Big(\{y_\nu\} + U_0(y_\nu + A_\nu\Big).$$

This defines $y_0,\ \ldots,\ \hat{y}_n$. We now show that (11) holds for $X = \{y_0,\ \ldots,\ \hat{y}_n\}$. First of all, $[X]^2 \subset K_1$ by definition of y_{ν_0}. Also, if $X \,\bar{\varepsilon}\, \Lambda_2$, then there is $\nu_1 < n$ such that $X \subset \Sigma\,(\nu < \nu_1)\,A_\nu$, and then $y_{\nu_1} \,\varepsilon\, X \subset \Sigma\,(\nu < \nu_1)\,A_\nu$, which contradicts the definition of y_{ν_1}. This proves Theorem 4.

8. Our last theorem will imply that the assertion of Theorem 4 is false if $|\Omega|$ is any singular infinite cardinal, provided we assume a version of the general continuum hypothesis.

THEOREM 5. *Let a be a singular infinite cardinal number and let B be a non-empty set of cardinals less than a such that $b \varepsilon B$ implies $2^b = b^+$, and let $a = \sup(b \varepsilon B) b$*†. *Then there is a set Ω of sets such that, if $A = \Sigma(X \varepsilon \Omega) X$, and Λ_2 denotes the set of all subsets of A which are of second Ω-category then* (i) $|\Omega| = a$; (ii) $A \varepsilon \Lambda_2$; (iii) $A \not\rightarrow (3, \Lambda_2)^2$.

Proof of Theorem 5. Let $b \varepsilon B$. Then $2^{b^+} < a$. For since $a' < a$, it follows that a is a limit cardinal, and hence $b < a$; $b^+ < a$, and there is $c \varepsilon B$ such that $b^+ < c$ Then $2^{b^+} \leqslant 2^c = c^+ < a$. Let m and n be the initial ordinals of cardinal a' and a respectively. Then there are cardinals $a_\mu < a$ such that $a = \Sigma(\mu < m) a_\mu$. There are $b_\mu \varepsilon B$ such that

$$a_\mu \leqslant b_\mu \quad (\mu < m).$$

By Theorem 2 there are graphs $\Gamma_\mu{}^* = (S_\mu{}^*, T_\mu{}^*)$, without triangles, such that

$$\phi(\Gamma_\mu{}^*) = \chi(\Gamma_\mu{}^*) = b_\mu{}^+ \ (\mu < m); \quad S_\mu{}^* S_\nu{}^* = \emptyset \ (\mu < \nu < m).$$

Let
$$\Omega = \Sigma(\mu < m)\{X : X \subset S_\mu{}^*; \ [X]^2 T_\mu{}^* = \emptyset\},$$

$$A = \Sigma(X \varepsilon \Omega) X.$$

Then
$$|\Omega| \leqslant \Sigma(\mu < m) 2^{b_\mu{}^+} \leqslant a|m| = a.$$

On the other hand, there is $f \varepsilon \Omega^A$ such that $x \varepsilon f(x)$, for $x \varepsilon A$. Then $\mu < m$; $\{x, y\} \varepsilon T_\mu{}^*$ imply $f(x) \neq f(y)$. Hence $\chi(\Gamma_\mu{}^*) \leqslant |\Omega|$;

$$a = \Sigma(\mu < m) a_\mu \leqslant \Sigma(\mu < m) b_\mu{}^+ = \Sigma(\mu < m)\chi(\Gamma_\mu{}^*) \leqslant |\Omega|\,|m|; \ a \leqslant |\Omega|.$$

Therefore (i) holds.

If $\Omega' \subset \Omega$; $A \subset \Sigma(X \varepsilon \Omega') X$, then there is $g \varepsilon (\Omega')^A$ such that $x \varepsilon g(x)$, for $x \varepsilon A$. Again, the relations $\mu < m$; $\{x, y\} \varepsilon T_\mu{}^*$ imply $g(x) \neq g(y)$, and hence we have $X(\Gamma_\mu{}^*) \leqslant |\Omega'|$;

$$a \leqslant \Sigma(\mu < m)\chi(\Gamma_\mu{}^*) \leqslant |\Omega'|\,|m|; \quad a \leqslant |\Omega'|.$$

This proves (ii).

We now consider the partition

$$[A]^2 = K_0 + K_1, \text{ where } K_0 = \Sigma(\mu < m)\Gamma_\mu{}^*; \ K_1 = [A]^2 - K_0.$$

If $Y \subset A$ and $[Y]^2 \subset K_0$, then $[Y]^2 \subset T_\mu{}^*$, for some $\mu < m$, and therefore, since $\Gamma_\mu{}^*$ does not contain any triangle, $|Y| < 3$.

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† Such a set B exists, for instance, if a is such that $\aleph_0 \leqslant b < a$ implies $2^b = b^+$, in which case we may take $B = \{b : \aleph_0 \leqslant b < a\}$.

On the other hand, if $Z \subset A$ and $[Z]^2 \subset K_1$, then $ZS_\mu^* \varepsilon \Omega$;

$$Z = \Sigma (\mu < m) ZS_\mu^* = \Sigma (X \varepsilon \Omega'') X,$$

where $\Omega'' = \{ZS_\mu^* : \mu < m\} \subset \Omega$; $|\Omega''| \leqslant |m| < a$. Hence $Z \notin \Lambda_2$, and (iii) follows. This completes the proof of Theorem 5.

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