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ON THE PRODUCT $\prod_{k=1}^{n} (1 - z^{a_k})$

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ON THE PRODUCT $\prod_{k=1}^{n}(1-z^{a_k})$.

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Consider the product $\prod_{k=1}^{n}(1-z^{a_k})$ where $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ are positive integers. Put

$$\max_{|z|=1} \Big|\prod_{i=1}^{n}(1-z^{a_i})\Big| = M(a_1, a_2, \ldots, a_n), \quad f(n) = \min_{a_1, a_2, \ldots, a_n} M(a_1, a_2, \ldots, a_n).$$

Clearly $M(a_1, \ldots, a_n) \leqslant 2^n$ (equality if and only if $(a_1, a_2, \ldots, a_n) > 1$ or $a_1 = a_2 = \cdots = a_n = 1$). The determination of $f(n)$ seems to be a very difficult question, and even a good estimation of $f(n)$ does not seem easy. In the present note we are going to prove that $f(n)^{1/n} \to 1$ as $n \to \infty$, and it seems possible that a refinement of our method would give ($\exp z = e^z$)

$$f(n) < \exp(n^{1-c})$$

for some $c < 1$. The lower bound $f(n) \geqslant \sqrt{2n}$ is nearly trivial, and we are unable at present to do any better.

We want to remark that it is easy to show that

$$\lim_{n=\infty} [M(1, 2, \ldots, n)]^{1/n}$$

exists and is between 1 and 2.

Put $z = e^{2\pi i \alpha}$, $<\alpha> = |1 - e^{2\pi i \alpha}|$. Several further questions can be asked. It is not difficult to prove that for almost all α (almost all means except a set of Lebesgue measure 0)

$$\underline{\lim} \prod_{k=1}^{n} < k\alpha > = 0. \tag{1}$$

We only outline the proof of (1). A special case of a well known theorem of Khintchine states that for almost all α there is an infinite sequence of integers p_n and q_n satisfying

$$(2)\qquad \left|\alpha-\frac{p_n}{q_n}\right| = o\left(\frac{1}{q_n^2\log q_n}\right).$$

A simple computation then shows that

$$\lim \prod_{k=1}^{q_n} <k\alpha> = 0.$$

Perhaps (1) holds for all α.

It is easy to see that

$$(3)\qquad \overline{\lim} \prod_{k=1}^{n} <k\alpha> = \infty.$$

holds for almost all α. (Clearly (3) can not hold for all α, e. g. it fails if α is rational). To see this we observe that a simple computation shows that if the q_n are the integers satisfying (2) then

$$\lim \prod_{k=1}^{q_n-1} <k\alpha> = \infty.$$

Perhaps one could determine how fast (1) tends to 0 and (3) tends to ∞ for almost all α.

Is it true that for all α

$$(4)\qquad \overline{\lim} \max_{|z|=1} \prod_{k=1}^{n} |z-e^{2\pi i k\alpha}| = \infty\,?$$

An old conjecture of P. Erdős which would imply (4) states as follows: Let $z_1, z_2, \ldots$ be any infinite sequence satisfying $|z_i|=1$.

Then

$$\overline{\lim} \max_{|z|=1} \prod_{i=1}^{n} |z-z_i| = \infty.$$

On the other hand a simple computation shows that for the q_n satisfying (2)

$$(5)\qquad \lim \max_{|z|=1} \prod_{k=1}^{q_n} |z-e^{2\pi i k\alpha}| = 2,$$

and perhaps

$$\underline{\lim} \max_{|z|=1} \prod_{k=1}^{n} |z-e^{2\pi i k\alpha}| < \infty$$

for all irrational α (it certainly is ∞ for rational α).

Finally we pose the following problem: Let $a_1 < a_2 < \ldots$ be any infinite sequence of integers. Is it true that for almost all α

$$\overline{\lim} \prod_{k=1}^{n} < a_k \alpha > = \infty, \ \underline{\lim} \prod_{k=1}^{n} < a_k \alpha > = 0?$$

Throughout this paper $0 \leqslant \alpha < 1$ and $c_1, c_2, \ldots$ will denote positive absolute constants, $|\theta| < 1$ (and the θ's appearing are not necessarily the same).

LEMMA 1. *Let*

$$\alpha = \frac{p}{q} + \frac{\theta}{q^2}, \quad (p, q) = 1. \tag{6}$$

Then for every l

$$\prod_{t=l+1}^{l+q} < t\alpha > < q^{a_1}. \tag{7}$$

If $\theta = 0$ the product in (7) is 0, hence (7) holds. Thus we can assume $\theta \neq 0$. Order the numbers $e^{2\pi i t\alpha}$, $l+1 \leqslant t \leqslant l+q$ according to the size of their arguments and denote them by $z_1, z_2, \ldots, z_q$ ($0 < \arg z_1 < \arg z_2 < \ldots < \arg z_q < 2\pi$, i. e. (6) implies that the z's are all different). From (6) we have

$$\arg z_k = 2\pi\left(\frac{k}{q} + \frac{\theta}{q}\right) \quad (k = 1, 2, \ldots, q). \tag{8}$$

Put $y_k = e^{2\pi i (k-1/2)/q}$. From (8) we evidently have

$$|1 - z_k| < |1 + y_k| \left(1 + c_2\left(\frac{1}{k} + \frac{1}{q+1-k}\right)\right). \tag{9}$$

Now from $\prod_{k=1}^{q} |1 - y_k| = 2$, $\left(\prod_{k=1}^{q} |1 - y_k|\right.$ is simply the value at $z = 1$ of $\left.(z^{2k} - 1)/(z^k - 1) = z^k + 1\right)$ and (9) we have

$$\prod_{k=1}^{q} |1 - z_k| < 2 \prod_{k=1}^{q} \left(1 + c_2\left(\frac{1}{k} + \frac{1}{q+1-k}\right)\right) < q^{c_1},$$

which proves the Lemma.

THEOREM 1. *To every* ε *there exists an* $n_0(\varepsilon)$ *and* $A = A(\varepsilon)$, $B = B(\varepsilon)$ *so that for every* $n > n_0(\varepsilon)$ *and every* α *which does not satisfy one of the* *inequalities*

$$(10) \qquad \frac{1}{Bn} < \left| \alpha - \frac{p}{q} \right| \leqslant \frac{1}{\varepsilon n} \qquad \textit{for some } 0 \leqslant p < q \leqslant A$$

we have

$$(11) \qquad \prod_{t=1}^{n} < t\alpha > < (1+\varepsilon)^n.$$

Theorem 1 means that (11) is satisfied except if α can be approximated „well" but not „too well" by rational fractions with „small" denominators.

Assume first that α is such that for every p and $q \leqslant A$

$$(12) \qquad \left| \alpha - \frac{p}{q} \right| \geqslant \frac{1}{\varepsilon n}.$$

By a well known theorem of Dirichlet there exists a $q \leqslant \varepsilon n$ for which

$$(13) \qquad \left| \alpha - \frac{p}{q} \right| < \frac{1}{q\varepsilon n} < \frac{\theta}{q^2} \qquad (p, q) = 1.$$

By (12) $q > A$. Put $uq \leqslant n < (u+1)q$. Then we have by $< t\alpha > \leqslant 2$ and our Lemma (since $2^{\varepsilon} < 1+\varepsilon$ for small ε and $q^{1/q} < A^{1/A}$ for $q > A > e$)

$$(14) \qquad \prod_{t=1}^{n} < t\alpha > < 2^q q^{c_1 u} < 2^{\varepsilon n} q^{c_1 \frac{n}{q}} < 2^{\varepsilon n} A^{c_1 \frac{n}{A}} < (1+\varepsilon)^n$$

if $A > A(\varepsilon)$ is large enough.

If for some $q \leqslant A$, $\left| \alpha - \frac{p}{q} \right| \leqslant \frac{1}{\varepsilon n}$ then by (10) we can assume that $\left| \alpha - \frac{p}{q} \right| < \frac{1}{Bn}$. But then the arguments of the numbers $e^{2\pi i(\nu q + l)\alpha}$, $\nu < u$, $1 \leqslant l \leqslant q$ $(uq \leqslant n < (u+1)q)$ differ from the corresponding q-th roots of unity by less than $1/B$. Thus for B sufficiently large a simple computation gives

$$\prod_{\nu q < l \leqslant (\nu+1)q} < t\alpha > < \frac{1}{2},$$

or

$$(15) \quad \prod_{t=1}^{n} < t\alpha > = < \prod_{t=1}^{uq} < t\alpha > \prod_{uq < t \leqslant n} < t\alpha > \ < \left(\frac{1}{2}\right)^u 2^q < \left(\frac{1}{2}\right)^{n/A} 2^A < 1$$

and Theorem 1, follows from (14) and (15).

Next we prove

THEOREM 2.

$$\lim f(n)^{1/n}=1.$$

Let $m^2 \leqslant n < (m+1)^2$. Consider the product

$$g_n(z)=\prod_{k=1}^{m}\prod_{l=1}^{m}(1-z^{2^k l})(1-z)^{n-m^2}.$$

In other words $a_1=a_2=\cdots=a_{n-m^2}=1$ and the other a's are the integers $2^k l$, $1\leqslant k\leqslant m$, $1\leqslant l\leqslant m$. To prove Theorem 2 it will be sufficient to show that

(16)
$$\lim \max_{|z|=1}|g_n(z)|^{1/n}=1.$$

We evidently have for $|z|\leqslant 1$

$$|1-z|^{n-m^2}\leqslant 2^{\,2\sqrt{n}} \quad (\text{i. e. } n-m^2\leqslant 2\sqrt{n}).$$

Thus to prove (16) it will suffice to show that for every ε if $m>m_0(\varepsilon)$

(17)
$$\max_{|z|=1}\prod_{k=1}^{m}\prod_{l=1}^{m}|1-z^{2^k l}|=\max_{0\leqslant\alpha\leqslant 1}\prod_{k=1}^{m}\prod_{l=1}^{m}<2^k l\alpha><(1+2\varepsilon)^{m^2}.$$

Consider the numbers $2^k\alpha=\alpha_k$, $1\leqslant k\leqslant m$. We claim that only $o(m)$ of them satisfy (10). Since $q\leqslant A$ it will suffice to show that only $o(m)$ of them satisfy (10) for a fixed q.

Suppose in fact that α_k satisfies (10) for a certain q. Then we have $\left|\alpha_k-\frac{p}{q}\right|=\frac{b_k}{n}$ where $\frac{1}{B}\leqslant|b_k|\leqslant\frac{1}{\varepsilon}$. Also $\left|\alpha_{k+1}-\frac{p'}{q}\right|=\frac{2b_k}{n}$ where $p'\equiv 2p$ (mod q). Thus (10) can be satisfied for at most $\frac{\log B/\varepsilon}{\log 2}+1$ consecutive values of k and these are followed by at least $c_3\log n$ values of k for which (10) is not satisfied for this particular value of q applying this argument for all the $k\leqslant m$ which satisfy (10) we obtain that (10) is satisfied for only $o(m)$ values of k, as stated.

Now we can prove (17). Write

$$\prod_{k=1}^{m}\prod_{l=1}^{m}<2^k l\alpha>=\prod_{k}{}_{1}\prod_{l=1}^{m}<2^k l\alpha>\prod_{k}{}_{2}\prod_{l=1}^{m}<2^k l\alpha>$$

where in Π_1 k is such that $2^k\alpha = \alpha_k$ satisfies (10). Clearly $\prod_{l=1}^{m} < 2^k l\alpha > \leqslant 2^m$, thus by what we just proved

$$\Pi_1 \prod_{k}^{m}_{l=1} < 2^k l\alpha > = 2^{o(m^2)}. \tag{18}$$

By Theorem 1 we have for every k in Π_2

$$\prod_{l=1}^{m} < 2^k l\alpha > < (1+\varepsilon)^m.$$

Thus

$$\Pi_2 \prod_{k}^{m}_{l=1} < 2^k l\alpha > < (1+\varepsilon)^{m^2}. \tag{19}$$

(18) and (19) implies (17), and thus Theorem 2 is proved.

THEOREM 3.

$$f(n) \geqslant \sqrt{2n}.$$

To prove Theorem 3 write

$$\prod_{i=1}^{n}(1-x^{a_i}) = \sum_i x^{b_i} - \sum_i x^{c_i}, \quad b_1 < b_2 < \ldots;\ c_1 < c_2 < \ldots$$

First we show that

$$\sum_i b_i^p = \sum_i c_i^p, \quad p = 0,1,\ldots, n-1. \tag{20}$$

To show (20) observe that 1 is an n-fold root of $\prod_{i=1}^{n}(1-x^{a_i})$. Thus $f^{(p)}(1) = 0$ for $p = 0, 1, \ldots, n-1$; or

$$\sum_i b_i(b_i-1)\ldots(b_i-p+1) = \sum_i c_i(c_i-1)\ldots(c_i-p+1),\ p = 0,1,\ldots,n-1,$$

which implies (20). From (20) we immediately obtain that at least n b's and n c's do not vanish which implies Theorem 3 by Parseval's equality.

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